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Unitary representations of $U_q(\mathfrak{sl}(2,\mathbb{R}))$, the modular double, and the multiparticle q-deformed Toda chains

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Abstract

The paper deals with the analytic theory of the quantum q-deformed Toda chains; the technique used combines the methods of representation theory and the Quantum Inverse Scattering Method. The key phenomenon which is under scrutiny is the role of the modular duality concept (first discovered by L.Faddeev) in the representation theory of noncompact semisimple quantum groups. Explicit formulae for the Whittaker vectors are presented in terms of the double sine functions and the wave functions of the N-particle q-deformed open Toda chain are given as a multiple integral of the Mellin-Barnes type. For the periodic chain the two dual Baxter equations are derived.

Preface

In the late seventies B. Kostant [1] has discovered a fascinating link between the representation theory of non-compact semisimple Lie groups and the quantum Toda chain. Let G be a real split semisimple Lie group, $B = MAN$ its minimal Borel subgroup, let N and $V = \bar{N}$ be the corresponding opposite unipotent subgroups. Let χ_N, χ_V be nondegenerate unitary characters of N and V, respectively. Let $\mathcal{H}_T$ be the space of smooth functions on G which satisfy the functional equation

$$\varphi(vxn) = \chi_V(v)\overline{\chi_N(n)}\,\varphi(x), \qquad v \in V, n \in N.$$

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A function $\varphi \in \mathcal{H}_T$ is uniquely determined by its restriction to $A \subset G$. Obviously, $\mathcal{H}_T$ is invariant under the action of the center of the universal enveloping algebra $Z \subset U(\mathfrak{g})$; hence, any Casimir operator $C \in Z$ gives rise to a differential operator acting in $C^\infty(A)$. When C is the quadratic Casimir, this is precisely the Toda Hamiltonian; other Casimirs provide a complete set of quantum integrals of motion.

This observation reduces the spectral theory of Toda chain to the representation theory of semisimple Lie groups. The joint eigenfunctions of the quantum Toda Hamiltonians are the so called generalized Whittaker functions. The theory of Whittaker functions has been extensively studied in the 60's and 70's [2], [3], [4]; it displays deep parallels with the celebrated Harish-Chandra theory of spherical functions [5] and depends on a profound study of the principal series representations [6].

The group theoretic approach based on representation theory of finite-dimensional semisimple groups is matched by a more sophisticated technique of the Quantum Inverse Scattering Method [7]. The treatment of the Toda chain by means of QISM is based on a 2×2 matrix first order difference Lax operator for the Toda lattice. (In order to understand its relation to the $n \times n$ Lax representation which is implicit in Kostant's approach recall that the Lax matrix is a tridiagonal Jacobi matrix which defines a three-term recurrence relation and hence may be regarded as a second order scalar difference operator). While the use of the lattice Lax representation restricts generality: we have to assume that $\mathfrak{g} = \mathfrak{sl}(n)$ [1], it allows to bring into play the powerful machinery of quantum R-matrices (and hence eventually of *infinite dimensional* quantum groups). Recently the first two authors have established an explicit connection of the QISM-based approach to the quantum Toda chain to the theory of Whittaker functions [9]. The technique of QISM yields new explicit formulae for the Whittaker functions which, to the best of our knowledge, were not known in the elementary representation theory.

It looks rather natural to generalize this approach to the q-deformed case. The use of lattice Lax representation makes the procedure rather straightforward: one simply has to replace the rational R-matrix with the trigonometric one (we shall see, however, that this generalization includes a number of nontrivial points). On the other hand, the very definitions of 'noncompact quantum groups' which one needs to proceed with the q-deformed version of the Kostant approach are by no means obvious. It is the interplay of the explicit formulae based on QISM and of their not-yet-defined counterparts coming from the representation theory of noncompact finite-dimensional quantum groups that makes the entire game very exciting. Our preliminary results suggest that the correct treatment of the problem requires a very significant change in the entire framework of the representation theory of $U_q(\mathfrak{g})$; the crucial role is played by the 'modular dual' of $U_q(\mathfrak{g})$ and the *modular double* $U_q(\mathfrak{g}) \otimes U_{\tilde{q}}(\mathfrak{g})$ which was introduced recently by Faddeev [10] [2]. Among other things, this new point of view leads to new possibilities in the choice of real forms of the relevant

[1] The treatment of other classical Lie algebras is also possible; for that end, one needs to use lattice Lax pairs with boundary conditions introduced by Sklyanin [8]. In the present note we shall not deal with this generalization and assume that $\mathfrak{g} = \mathfrak{sl}(n)$

[2] The definition of the modular double was coined out by Faddeev in the special case $\mathfrak{g} = \mathfrak{sl}(2)$; as pointed out to the authors by B. Feigin, it is most likely that for general semisimple Lie algebras the modular dual of $U_q(\mathfrak{g})$ is $U_{\tilde{q}}(\check{\mathfrak{g}})$, where $\check{\mathfrak{g}}$ is the Langlands dual of $\mathfrak{g}$.

algebras: it is the real form of the modular double $U_q(\mathfrak{g}) \otimes U_{\tilde{q}}(\mathfrak{g})$ which really matters. One nontrivial possibility for the choice of the real form has been recently pointed out by Faddeev, Kashaev and Volkov [11] in their study of the quantum Liouville theory; it is very encouraging that the same real form naturally arises in the study of the q-deformed Toda chain.

Analytical aspects of the theory bring into play the double gamma and double sine functions of Barnes [12, 13, 14, 15, 16], or the closely related quantum dilogarithms [17], which replace the ordinary gamma functions in the formulae for both the Harish-Chandra c-functions and the Whittaker functions. We believe that the implications of these constructions for the representation theory are probably more interesting than the q-deformed Toda model itself (commonly known as the relativistic Toda chain [18]).

Our strategy in the present paper is as follows. In Section 1 we shall start with the elementary representation theory of the algebra $U_q(\mathfrak{sl}(2,\mathbb{R}))$. Section 2 deals with the theory of Whittaker vectors and Whittaker functions for the modular double $U_q(\mathfrak{sl}(2,\mathbb{R}))\otimes U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ and with the 2-particle q-deformed open Toda chain. We obtain explicit formulae for the Whittaker vectors in terms of the double sine functions and derive the integral representations for solutions to one-parameter family of two-particle relativistic Toda chains in the framework of representation theory; all these solutions enjoy with the dual symmetry. Generalization to the N-particle case is described in Section 3; using the QISM approach, we derive an appropriate solution to the spectral problem for the open N-particle chain in the form of multiple integral of the Mellin-Barnes type with the natural deformation of the usual gamma functions to the double sine functions. It is shown that the solution for the N-periodic chain is represented as a generalized Fourier transform of $N-1$-particle open wave function with the kernel satisfying two mutually dual Baxter equations. Finally, in Appendix we list the essential analytic properties of the double sine functions.

1 Representations of principal series of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ and the Modular Double

In this section we shall discuss the representations of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ which may be regarded as deformations of the principal series representations of $SL(2,\mathbb{R})$. As pointed out by Faddeev [10], these representations possess a remarkable duality which is similar to the modular duality for noncommutative tori discovered by Rieffel [19]. We start with the algebraic definition of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ (see, for example, [20]). It is generated by elements $K^{\pm 1}, E, F$ subject to the relations

$$\begin{gathered} KE = q^2 EK \qquad KF = q^{-2} FK, \\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \end{gathered} \tag{1.1}$$

where

$$q = e^{\pi i \tau}, \qquad \tau \in \mathbb{C} \tag{1.2}$$

The bialgebra structure on $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is given by the coproduct [3]

$$\begin{aligned} \Delta K &= K \otimes K, \\ \Delta E &= E \otimes 1 + K \otimes E, \\ \Delta F &= 1 \otimes F + F \otimes K^{-1}. \end{aligned} \tag{1.3}$$

The center of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is generated by the Casimir element

$$C_2 = qK + q^{-1}K^{-1} + (q-q^{-1})^2 FE. \tag{1.4}$$

The algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ admits a real form defined by the involution

$$K^* = K, \quad E^* = -E, \quad F^* = -F, \tag{1.5}$$

which is compatible with the commutation relations (1.1) only if $|q| = 1$, i.e. $\tau \in \mathbb{R}$. The corresponding real algebra is called $U_q(\mathfrak{sl}(2,\mathbb{R}))$. (We shall see later that when $U_q(\mathfrak{sl}(2))$ is replaced with its modular double, there is a possibility to choose the real structure in a different way.)

It is sometimes useful to consider the corresponding 'infinitesimal' algebra $U_\tau(\mathfrak{sl}(2,\mathbb{R}))$, $\tau \in \mathbb{R}$ with generators $E,\ F,\ H$ and relations

$$\begin{aligned} &[H,\, E] = 2E, \qquad [H,\, F] = -2F, \\ &[E,\, F] = \frac{q^H - q^{-H}}{q - q^{-1}}. \end{aligned} \tag{1.6}$$

Evidently, there is an involution

$$H^* = -H, \quad E^* = -E, \quad F^* = -F. \tag{1.7}$$

Let us sketch the representation theory of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ in the way which stresses the role of the modular duality concept (cf. [21]). The representations of the principal series of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ admit an explicit realization by means of finite difference operators on the real line; the commutation relations of the basic operators which are the building blocks for these representations are the ordinary Weyl relations. To put it in a different way, the principal series representations of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ factor through a noncommutative torus.

Definition 1.1 *The noncommutative torus $\mathbb{A}_q$ is the associative algebra generated by u, v subject to the relation $uv = q^2 vu$.*

We shall adjoin to $\mathbb{A}_q$ the inverse elements u^{-1}, v^{-1}, (in other words, we replace $\mathbb{A}_q$ with its field of fractions, which we denote by the same letter).

Proposition 1.1 *For any $z \in \mathbb{C}$ the mapping $U_q(\mathfrak{sl}(2,\mathbb{R})) \to \mathbb{A}_q$ defined by*

$$K \mapsto zu^{-1}, \quad E \mapsto \frac{v^{-1}}{q-q^{-1}}(1-u^{-1}), \quad F \mapsto \frac{qv}{q-q^{-1}}(z - z^{-1}u) \tag{1.8}$$

is a homomorphism of algebras.

[3] The coalgebraic structure of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is not used in the present paper.

Note that the Casimir C_2 is mapped by the homomorphism (1.8) to $qz + q^{-1}z^{-1}$.

It is sometimes technically convenient to extend the algebra $U_q(\mathfrak{sl}(2))$ by adjoining to it 'virtual Casimir elements'. The following assertion is well-known.

Proposition 1.2 *The center of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ is isomorphic to the polynomial algebra $Z = \mathbb{C}[qz + q^{-1}z^{-1}] \subset \mathbb{C}[z, z^{-1}] = \hat{Z}$; $U_q(\mathfrak{sl}(2,\mathbb{R}))$ is a free Z-module.*

Set

$$\hat{U}_q(\mathfrak{sl}(2,\mathbb{R})) = U_q(\mathfrak{sl}(2,\mathbb{R})) \otimes_Z \hat{Z}.$$

The mapping (1.8) canonically extends to $\hat{U}_q(\mathfrak{sl}(2))$. Informally, we may think of $\hat{U}_q(\mathfrak{sl}(2))$ as of a bundle of noncommutative tori parameterized by the spectrum of the central element $z \in \hat{Z}$.

Proposition 1.1 is a simple instance of the 'free field representations' for quantum groups; it may also be compared with the well-known Gelfand-Kirillov [22] theorem which asserts that the field of fractions of the universal enveloping algebra is isomorphic to the standard noncommutative division algebra (central extension of the field of fractions of the Weyl algebra generated by several pairs of 'canonical variables' p_i, q_i).

As a motivation for the study of the modular duality for $U_q(\mathfrak{sl}(2,\mathbb{R}))$ let us recall the following standard construction from ergodic theory [19], [10]. Let $q = \exp \pi i\omega_1/\omega_2$, where $\omega_1, \omega_2 \in \mathbb{R}$; we shall assume that $\tau = \omega_1/\omega_2$ is irrational. Put $\widetilde{q} = \exp(\pi i\omega_2/\omega_1)$ and let $\mathbb{A}_{\widetilde{q}}$ be the dual torus with generators $\widetilde{u}$, $\widetilde{v}$, and relations $\widetilde{u}\widetilde{v} = \widetilde{q}^{\,2}\widetilde{v}\widetilde{u}$. Let us define unitary operators $T_{\omega_1}, T_{\omega_2}, S_{-i\omega_1}, S_{-i\omega_2}$ in $L_2(\mathbb{R})$ by

$$\begin{aligned} T_{\omega_1}\varphi(t) &= \varphi(t+\omega_1), & T_{\omega_2}\varphi(t) &= \varphi(t+\omega_2), \\ S_{-i\omega_1}\varphi(t) &= e^{\frac{2\pi it}{\omega_1}}\varphi(t), & S_{-i\omega_2}\varphi(t) &= e^{\frac{2\pi it}{\omega_2}}\varphi(t). \end{aligned} \tag{1.9}$$

Define the dual representations of $\mathbb{A}_q$ and $\mathbb{A}_{\widetilde{q}}$ in $\mathcal{H} = L_2(\mathbb{R})$ by

$$\begin{aligned} \rho: &\quad u \mapsto T_{\omega_1}, & v &\mapsto S_{-i\omega_2}, \\ \widetilde{\rho}: &\quad \widetilde{u} \mapsto T_{\omega_2}, & \widetilde{v} &\mapsto S_{-i\omega_1}. \end{aligned} \tag{1.10}$$

It is easy to see that $\mathbb{A}_q$, and $\mathbb{A}_{\widetilde{q}}$ are the centralizers of each other in the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators in $\mathcal{H}$. The space $\mathcal{H} = L_2(\mathbb{R})$, which has the structure of a left $\mathbb{A}_q$-module and of a right $\mathbb{A}_{\widetilde{q}}$-module is called the *imprimitivity* $(\mathbb{A}_q,\ \mathbb{A}_{\widetilde{q}})$*-bimodule.* The images of $\mathbb{A}_q$, $\mathbb{A}_{\widetilde{q}}$ in $\mathcal{B}(\mathcal{H})$ are factors of type II_1. Clearly, the representations of $\mathbb{A}_q$ and $\mathbb{A}_{\widetilde{q}}$ are reducible (in fact, both $\mathbb{A}_q$ and $\mathbb{A}_{\widetilde{q}}$ contain plenty of idempotent elements which are represented by projection operators in $\mathcal{H}$; the image of a projection operator $\widetilde{P} \in \widetilde{\rho}(\mathbb{A}_{\widetilde{q}})$ is an invariant subspace for $\mathbb{A}_q$; the subspaces of $\mathcal{H}$ which arise in this way are the celebrated fractional dimensional spaces of von Neumann). On the other hand, the second commutant of $\mathbb{A}_q \otimes \mathbb{A}_{\widetilde{q}}$ coincides with $\mathcal{B}(\mathcal{H})$ and hence (1.10) is an irreducible representation of $\mathbb{A}_q \otimes \mathbb{A}_{\widetilde{q}}$ (as a matter of fact, up to unitary equivalence, this algebra has a unique irreducible representation).

The relation between the two noncommutative tori described above is called by Rieffel the *strong Morita equivalence;* In a more general way, Rieffel showed [19] that two tori $\mathbb{A}_q$ and $\mathbb{A}_{\widetilde{q}}$, $q = e^{\pi i \tau}, \widetilde{q} = e^{\pi i \widetilde{\tau}}$ are strong Morita equivalent if and only if $\widetilde{\tau} = \frac{a\tau + b}{c\tau + d}$, where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}),$$

which explains the term 'modular duality'.

Definition 1.2 *The modular dual of* $U_q(\mathfrak{sl}(2,\mathbb{R}))$ *is the Hopf algebra* $U_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ *with* $\widetilde{q} = e^{\pi i/\tau}$ *; we set also*

$$\hat{U}_{\widetilde{q}}\left(\mathfrak{sl}(2)\right) = U_{\widetilde{q}}\left(\mathfrak{sl}(2)\right) \otimes_{\widetilde{Z}} \hat{Z},$$
$$\widetilde{Z} = \mathbb{C}[\widetilde{q}z + \widetilde{q}^{\,-1}z^{-1}], \qquad \hat{Z} = \mathbb{C}[z, z^{-1}].$$

The obvious motivation for this definition is the existence of the 'dual free field representation' $U_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R})) \to \mathbb{A}_{\widetilde{q}}$.

Remark 1.1 *The modular transformation usually considered in the theory of theta functions is* $\tau \mapsto -\frac{1}{\tau}$ *; this transformation preserves the upper half-plane* $\operatorname{Im}\tau > 0$ *and the unit circle* $|q| < 1$*. While the flip* $\widetilde{q} \mapsto \widetilde{q}^{\,-1}$ *amounts to the simple exchange of the generators of the quantum torus (and hence, in particular, the quantum algebras* $U_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ *and* $U_{\widetilde{q}^{-1}}(\mathfrak{sl}(2,\mathbb{R}))$ *are isomorphic), our choice of the sign of the modular transform appears to be most natural for the study of* q*-deformed Toda chain.*

The fundamental difference which arises in the representation theory of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ is that its unitary representations are constructed from *non-unitary* representations of the quantum torus: we need to make a kind of 'Wick rotation' and hence $u, v \in \mathbb{A}_q$ are represented by *unbounded* operators [23]. More precisely, let us consider the following operators in $\mathcal{H}$:

$$\begin{aligned} &T_{i\omega_1}\varphi(t) = \varphi(t + i\omega_1), && T_{i\omega_2}\varphi(t) = \varphi(t + i\omega_2), \\ &S_{\omega_1}\varphi(t) = e^{\frac{2\pi t}{\omega_1}}\varphi(t), && S_{\omega_2}\varphi(t) = e^{\frac{2\pi t}{\omega_2}}\varphi(t). \end{aligned} \tag{1.11}$$

The dual representations of $\mathbb{A}_q$, $\mathbb{A}_{\widetilde{q}}$ are now given by

$$\begin{aligned} &\rho_W: \quad u \mapsto T_{i\omega_1}, \qquad v \mapsto S_{\omega_2}, \\ &\widetilde{\rho}_W: \quad \widetilde{u} \mapsto T_{i\omega_2}, \qquad \widetilde{v} \mapsto S_{\omega_1}. \end{aligned} \tag{1.12}$$

Operators (1.11) are essentially self-adjoint on the common domain $\mathcal{P}$ which consists of entire functions ψ such that

$$\int\limits_{\mathbb{R}} e^{sx}\left|\psi\left(x + iy\right)\right|^2 dx < \infty \qquad \text{for all}\, y \in \mathbb{R}, \quad s \in \mathbb{R}.$$

Remark 1.2 *Unlike the unitary case, the definition of the centralizer of an unbounded operator must take care of the domains of operators; thus* $AB = BA$ *implies that* $B(\mathrm{Dom}_A) \subset \mathrm{Dom}_A$*; this may be not true even if* B *is bounded. As a result, although the four operators (1.11) commute with each other, the same is not true, e. g., for their spectral projection operators* [4]*; thus, contrary to the ergodic case, the centralizer of* $\rho_W(\mathbb{A}_q)$ *does not contain projection operators, and hence the representations* $\rho_W, \widetilde{\rho}_W$ *are geometrically irreducible. It is much more important for us, however, that they are* not irreducible *in the operator sense, as each of them still admits a huge algebra of intertwiners.*

Proposition 1.3 *Operators which commute with all four operators (1.11) are scalars.*

Corollary 1.1 *Representation of* $\mathbb{A}_q \otimes \mathbb{A}_{\widetilde{q}}$ *is strongly irreducible (i.e. it does not admit any nontrivial intertwiners).*

Let us now describe explicitly the *particular* principal series representation of $\hat{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ which is extensively used in the paper (the realization we use is slightly different from those described in [21]). Namely, the representation π_λ of $\hat{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ ($q = e^{\pi i\omega_1/\omega_2}, \omega_1, \omega_2 \in \mathbb{R}_+$), which depends on a parameter $\lambda \in \mathbb{C}$, is given by

$$\pi_\lambda : \quad \begin{aligned} K &\mapsto e^{\frac{\pi i\lambda}{\omega_2}} T_{i\omega_1}^{-1}, \\ E &\mapsto \frac{S_{\omega_2}^{-1}}{q-q^{-1}}\left(1 - T_{i\omega_1}^{-1}\right), \\ F &\mapsto \frac{qS_{\omega_2}}{q-q^{-1}}\left(e^{\frac{\pi i\lambda}{\omega_2}} - e^{-\frac{\pi i\lambda}{\omega_2}} T_{i\omega_1}\right), \\ C_2 &\mapsto qe^{\frac{\pi i\lambda}{\omega_2}} + q^{-1}e^{-\frac{\pi i\lambda}{\omega_2}}, \\ z &\mapsto e^{\frac{\pi i\lambda}{\omega_2}}. \end{aligned} \tag{1.13}$$

By duality, we define the representation $\widetilde{\pi}_\lambda$ of the modular dual algebra $\hat{U}_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ with $\widetilde{q} = e^{\pi i\omega_2/\omega_1}$ by

$$\widetilde{\pi}_\lambda : \quad \begin{aligned} \widetilde{K} &\mapsto e^{\frac{\pi i\lambda}{\omega_1}} T_{i\omega_2}^{-1}, \\ \widetilde{E} &\mapsto \frac{S_{\omega_1}^{-1}}{\widetilde{q}-\widetilde{q}^{-1}}\left(1 - T_{i\omega_2}^{-1}\right), \\ \widetilde{F} &\mapsto \frac{\widetilde{q}\, S_{\omega_1}}{\widetilde{q}-\widetilde{q}^{-1}}\left(e^{\frac{\pi i\lambda}{\omega_1}} - e^{-\frac{\pi i\lambda}{\omega_1}} T_{i\omega_2}\right), \\ \widetilde{C}_2 &\mapsto \widetilde{q}\, e^{\frac{\pi i\lambda}{\omega_1}} + \widetilde{q}^{-1}e^{-\frac{\pi i\lambda}{\omega_1}}, \\ \widetilde{z} &\mapsto e^{\frac{\pi i\lambda}{\omega_1}}. \end{aligned} \tag{1.14}$$

The representations π_λ, $\widetilde{\pi}_\lambda$ are defined on a larger space $\mathcal{P}_\lambda \supset \mathcal{P}$ which depends on λ.

[4] Spectral projection operators $E(\Delta)$ for multiplication operators are multiplication operators by the characteristic function of the interval Δ; this function has compact support and hence the spectral projector does not preserve the domain which consists of analytic functions.

Definition 1.3 $\mathcal{P}_\lambda$ *is the set of entire functions such that*

(i). For $t \to +\infty$ *a function* $\psi \in \mathcal{P}_\lambda$ *admits an asymptotic expansion*

$$\psi(t+is) \sim_{t\to+\infty} e^{\frac{2\pi\lambda t}{\omega_1\omega_2}} \sum_{n_1,n_2\geq 0} C_{n_1,n_2} e^{\frac{-2\pi t(n_1\omega_1+n_2\omega_2)}{\omega_1\omega_2}} \tag{1.15}$$

uniformly in each bounded strip.

(ii). For $t \to -\infty$ *it admits an asymptotic expansion*

$$\psi(t+is) \sim_{t\to-\infty} C\Big(1 + \sum_{n_1,n_2>0} C_{n_1,n_2} e^{\frac{2\pi t(n_1\omega_1+n_2\omega_2)}{\omega_1\omega_2}}\Big) \tag{1.16}$$

uniformly in each bounded strip [5] *.*

The scalar product which is adapted to the discussion of the unitarity conditions in our algebra, defined on $\mathcal{P}_\lambda$ only for $\lambda \in i\mathbb{R} - \omega_1 - \omega_2$, is given by

$$(\varphi, \psi) = \int_{\mathbb{R}} e^{2\pi t\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)} \overline{\varphi(t)}\psi(t)dt \tag{1.17}$$

Proposition 1.4 *The following statements hold:*

(i) Operators $\pi_\lambda(X)$, $X \in U_q(\mathfrak{sl}(2,\mathbb{R}))$ *and* $\widetilde{\pi}_\lambda(Y)$, $Y \in U_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ *leave* $\mathcal{P}_\lambda \subset L_2(\mathbb{R})$ *invariant and commute with each other on this domain; any operator which commutes with both algebras is scalar.*

(ii) If $\lambda \in i\mathbb{R} - \omega_1 - \omega_2$ *, all operators* $\pi_\lambda(X)$, $X \in U_q(\mathfrak{sl}(2,\mathbb{R}))$ *obey the involution generated by (1.5) with respect to the scalar product (1.17). Similar statement holds for all operators* $\widetilde{\pi}_\lambda(Y)$, $Y \in U_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ *.*

(iii) The central characters $z, \widetilde{z}$ *of* π_λ, $\widetilde{\pi}_\lambda$ *are related by* $z = \widetilde{z}^{\,\tau}$, $\tau = \omega_1/\omega_2$ *.*

As in Proposition 1.3, commutativity condition implies that the operator preserves the domains of our unbounded operators.

Corollary 1.2 *The principal series representation* π_λ *of* $\hat{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ *canonically extends to a representation of* $\hat{U}_q(\mathfrak{sl}(2,\mathbb{R})) \otimes \hat{U}_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ *which is defined on the same domain* $\mathcal{P}_\lambda$ *; this representation is unitary if and only if* $\lambda \in i\mathbb{R} - \omega_1 - \omega_2$ *.*

Remark 1.3 *Since representations of the dual algebras* $U_q(\mathfrak{sl}(2))$ *and* $U_{\widetilde{q}}(\mathfrak{sl}(2))$ *are constructed from the dual representations of the quantum tori* $\mathbb{A}_q$, $\mathbb{A}_{\widetilde{q}}$ *for any values of the central characters* $z = e^{\pi i\lambda/\omega_2}, \widetilde{z} = e^{\pi i\mu/\omega_1}$ *, one might conclude that the principal series representations* π_λ, $\widetilde{\pi}_\mu$ *centralize each other for any pair of indices* λ, μ *; it is the condition on the common domain which imposes the selection rule.*

[5] The space $\mathcal{P}_\lambda$ essentially coincides with those considered in [21].

Let us now introduce the following key definition.

Definition 1.4 *The modular double of* $\hat{U}_q(\mathfrak{sl}(2))$ *is the Hopf algebra*

$$\mathcal{D}_{mod} = \hat{U}_q(\mathfrak{sl}(2)) \otimes \hat{U}_{\widetilde{q}}(\mathfrak{sl}(2)).$$

The bialgebra structure of the modular double, i.e., its product and coproduct, is standard. The point is that this algebra admits an unexpected class of representations which are *not* tensor products of representations of the factors, but rather are related to a kind of "type II" operator algebras (the quotation marks reflect the fact that due to analyticity constraints our algebras are "thinner" than the genuine type II factors; in particular, they do not contain projection operators). The modular double should be regarded as an analytic rather than algebraic object which for the first time brings into play the nontrivial analytic properties of noncompact semisimple quantum groups.

In what follows we shall be interested only in the *principal series representations* of $\mathcal{D}_{mod}$ defined above; with respect to this subclass of representations $\mathcal{D}_{mod}$ behaves itself as a rank one algebra. Note that the kernel of these representations contains the two-sided ideal $\mathcal{J} \subset \hat{U}_q(\mathfrak{sl}(2)) \otimes \hat{U}_{\widetilde{q}}(\mathfrak{sl}(2))$ generated by the relations

$$z = \widetilde{z}^{\,\tau}, \quad K = \widetilde{K}^{\,\tau}.$$

The use of the modular double and its representations, instead of those of its factors, appears to be very natural in many ways. We shall see below that the definition of the Whittaker vectors becomes unambiguous only if we require that they are the eigenvectors of *both* nilpotent generators $\pi_\lambda(E),\ \widetilde{\pi}_\lambda(\widetilde{E})$. One more reason to enjoy the presence of a double set of generators is the integrability problem for the q-deformed (relativistic) Toda model discussed below. The q-Toda Hamiltonian, which is derived from the Casimir element of $U_q(\mathfrak{sl}(2))$ is a difference operator which involves only translations $T_{i\omega_1}$; due to the presence of quasiconstants (i.e., functions with period $i\omega_1$), its spectrum becomes multiple with infinite multiplicity; the multiplicity problem is resolved when we take into account the dual Casimir element which involves dual translations $T_{i\omega_2}$.

The real form of $\mathcal{D}_{mod}$ used above is inherited from the real forms of $\hat{U}_q(\mathfrak{sl}(2)),\ \hat{U}_{\widetilde{q}}(\mathfrak{sl}(2))$. Interestingly, there exists a kind of complexification of the picture described above. As pointed out by Faddeev [10], for a special choice of the *complex* periods $\omega_1,\ \omega_2$ there exists another real form of $\mathcal{D}_{mod}$ which does not reduce to real forms of its factors. Namely,

Proposition 1.5 *(i) Let us assume that* $\omega_1 = \overline{\omega}_2$, *or, equivalently, that* $|\tau| = 1$. *Then the mapping*

$$E \mapsto -\widetilde{E},\ F \mapsto -\widetilde{F},\ K \mapsto \widetilde{K},\ z \mapsto \widetilde{q}^{\,2}\widetilde{z} \tag{1.18}$$

extends to a $\mathbb{C}$*-antilinear involution of* $\mathcal{D}_{mod}$.

(ii) Let ρ *be a unitary representation of* $\mathcal{D}_{mod}$ *with respect to the real form (1.18); then all operators* $\rho(X), X \in \hat{U}_q(\mathfrak{sl}(2)) \subset \mathcal{D}_{mod},\ \rho(Y), Y \in \hat{U}_{\widetilde{q}}(\mathfrak{sl}(2)) \subset \mathcal{D}_{mod}$, *are normal.*

(iii) Let $\lambda \in i\mathbb{R} - \omega_1 - \omega_2$; then the principal series representation π_λ extends to a unitary representation of $\mathcal{D}_{mod}$ with respect to the real form (1.18).

Physical self-adjoint Hamiltonians associated with the real form (1.18) can be derived from the real and imaginary parts of the Casimir operators. Analytically, the Faddeev's real form is particularly attractive, since in that case the lattice generated by ω_1, ω_2 is non-degenerate.

The use of the modular double is very well suited for the treatment of interpolation problems. Recall that we are dealing with the 'rational form' of the quantum algebra $U_q(\mathfrak{sl}(2))$ which is defined in terms of the generator $K = q^H$. This choice is at the core of the modular duality: it will be completely destroyed if we replace $U_q(\mathfrak{sl}(2))$ with the 'infinitesimal' algebra $U_\tau(\mathfrak{sl}(2))$ generated by E, F, H , the commutativity of two dual sets of generators will be destroyed. On the other hand, when it comes up to compute special functions associated with representations of $U_q(\mathfrak{sl}(2))$, i.e., some specific matrix coefficients of its irreducible representations, e.g., spherical functions or Whittaker functions, and to construct the corresponding spectral theory, it is important to define these functions on the entire real line or on its compexification. By contrast, the use of the rational form $U_q(\mathfrak{sl}(2))$ implies that these functions are defined *a priori* only on a discrete set $\{K^n,\, n \in \mathbb{Z}\}$. Let assume that $\omega_1,\ \omega_2$ are real and $\tau = \omega_1/\omega_2$ is irrational.

Proposition 1.6 *For any $\alpha \in \mathbb{R}$ operators $\pi_\lambda(e^{\alpha H})$ are approximated by linear combinations of $\pi_\lambda(K^n \cdot \widetilde{K}^m),\ n, m \in \mathbb{Z}$.*

Indeed, $\pi_\lambda(e^{\alpha H})$ is a translation operator, $\pi_\lambda(e^{\alpha H})\varphi(t) = \varphi(t - i\alpha)$; on the other hand, $\pi_\lambda(K^n \cdot \widetilde{K}^m)\varphi(t) = e^{\pi i\lambda n/\omega_2 + \pi i\lambda m/\omega_1}\varphi(t - in\omega_1 - im\omega_2)$. The set $\{in\omega_1 + im\omega_2\ ;\ n, m \in \mathbb{Z}\}$ is dense in $i\mathbb{R}$.

Remark 1.4 *There exists a whole family of principal series representations similar to those described above. It is easy to find a realization of the algebras $U_q(\mathfrak{sl}(2))$ and $U_{\widetilde{q}}(\mathfrak{sl}(2))$ labeled by integer indices k_1 and k_2 , respectively, such that all representation operators, which act on an appropriate space $\mathcal{P}_\lambda^{k_1 k_2}$, satisfy the unitarity condition with respect to the scalar product with the measure $\exp\{\frac{2\pi(k_1\omega_1 + k_2\omega_2)t}{\omega_1\omega_2}\}$. In this case one obtains unitary representation if and only if $\lambda \in i\mathbb{R} - k_1\omega_1 - k_2\omega_2$. For simplicity, in the present paper we restrict ourself to the case $k_1 = k_2 = 1$, although the more general case can be treated quite similarly.*

2 Whittaker vectors

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{n}$ its maximal nilpotent subalgebra generated by positive root vectors. A character $\chi : \mathfrak{n} \to \mathbb{C}$ is uniquely fixed by its values on root vectors associated with simple roots; it is called nondegenerate if $\chi(e_\alpha) \neq 0$ for all simple roots α . A Whittaker vector in a $\mathfrak{g}$-module V is a vector $w \in V$ such that

$$Xw = \chi(X)w \tag{2.1}$$

for all $X \in \mathfrak{n}$ The extension of this definition to q-deformed algebras is nontrivial: it is easy to see that for rank $\mathfrak{g} \geq 2$ the algebra $U_q(\mathfrak{n})$ generated by the Chevalley generators associated with positive simple roots does not admit nondegenerate characters (the obstruction is associated with the q-deformed Serre relations). In [24] Sevostyanov found the way around this difficulty: one has to rescale the generators of the nilpotent subalgebra multiplying them by appropriate group-like elements from the Cartan subalgebra. Although the Serre relations are vacuous in the $\mathfrak{sl}_2$ case, the same trick proves worthy in that case as well; it provides an extra freedom which serves to construct various versions of the q-deformed Toda Hamiltonians.

Whittaker vectors associated with the unitary principal series representations of $SL(2,\mathbb{R})$ do not lie in the Hilbert space, because the spectrum of E, F is continuous; as a result, the Whittaker functions which are defined as formal matrix coefficients of the principal series representations between a pair of Whittaker vectors are expressed by a divergent integral which requires regularization. The situation in the q-deformed case is completely similar. As already mentioned, the natural definition of Whittaker vectors in the q-deformed setting requires the use of the modular double. The two commuting generators $E,\ \widetilde{E} \in \mathcal{D}_{mod}$ give rise to two compatible difference equations which have a unique common solution with nice analytic properties; this solution does not belong to $L_2(\mathbb{R})$, because it does not decrease rapidly enough.

With these remarks in mind, we may now proceed to the formal definition. Let $(\pi_\lambda, \widetilde{\pi}_\lambda)$, $\lambda \in i\mathbb{R} - \omega_1 - \omega_2$ be the unitary representation of $\mathcal{D}_{mod}$ and $\alpha \in \mathbb{R}$ be an arbitrary parameter. The E-Whittaker vector $\Phi_\lambda^{(\alpha)}$ is defined by

$$\begin{aligned} \pi_\lambda(E)\Phi_\lambda^{(\alpha)} &= \frac{g^{\omega_1}}{q-q^{-1}}\, e^{\pi i\alpha}\, \pi_\lambda(q^{\alpha H})\Phi_\lambda^{(\alpha)}, \\ \widetilde{\pi}_\lambda(\widetilde{E})\Phi_\lambda^{(\alpha)} &= \frac{g^{\omega_2}}{\widetilde{q}-\widetilde{q}^{-1}}\, e^{\pi i\alpha}\, \widetilde{\pi}_\lambda(\widetilde{q}^{\,\alpha\widetilde{H}})\Phi_\lambda^{(\alpha)}; \end{aligned} \tag{2.2}$$

here g is a positive real number (the 'coupling constant'). The extra parameter α matches the freedom in the choice of the quantum Lax operator in the alternative formulation of the q-deformed Toda theory based on Quantum Inverse Scattering Method. In other words, particular choices of α correspond to different Toda-like models.

In a similar way, the F-Whittaker vectors are defined by

$$\begin{aligned} \pi_\lambda(F)\widehat{\Phi}_\lambda^{(\alpha)} &= -\frac{g^{\omega_1}}{q-q^{-1}}\, e^{\pi i\alpha}\, \pi_\lambda(q^{-\alpha H})\widehat{\Phi}_\lambda^{(\alpha)}, \\ \widetilde{\pi}_\lambda(\widetilde{F})\widehat{\Phi}_\lambda^{(\alpha)} &= -\frac{g^{\omega_2}}{\widetilde{q}-\widetilde{q}^{-1}}\, e^{\pi i\alpha}\, \widetilde{\pi}_\lambda(\widetilde{q}^{\,-\alpha\widetilde{H}})\widehat{\Phi}_\lambda^{(\alpha)}. \end{aligned} \tag{2.3}$$

The definition of the Whittaker vectors is completely symmetric with respect to the exchange of the two dual algebras $U_q(\mathfrak{sl}(2)),\ U_{\widetilde{q}}(\mathfrak{sl}(2))$. Note that the existence of a common eigenvector of the commuting generators $\pi_\lambda(E),\ \widetilde{\pi}_\lambda(\widetilde{E})$, or $\pi_\lambda(F),\ \widetilde{\pi}_\lambda(\widetilde{F})$ is guaranteed due to our 'selection rule' for the central characters of $\hat{U}_q(\mathfrak{sl}(2)),\ \hat{U}_{\widetilde{q}}(\mathfrak{sl}(2))$.

2.1 Whittaker vectors: Explicit solutions

We shall start with the explicit formulae for the simplest Whittaker vectors corresponding to a particular choice of α. Using the representations (1.13), (1.14), we get the following system of difference equations for the vectors $\Phi_\lambda^{(\alpha)}$ with $\alpha = 0, 1$:

$$\frac{\Phi_\lambda^{(0)}(t - i\omega_1)}{\Phi_\lambda^{(0)}(t)} = 1 - g^{\omega_1} e^{\frac{2\pi t}{\omega_2}}, \tag{2.4a}$$

$$\frac{\Phi_\lambda^{(0)}(t - i\omega_2)}{\Phi_\lambda^{(0)}(t)} = 1 - g^{\omega_2} e^{\frac{2\pi t}{\omega_1}}, \tag{2.4b}$$

$$\frac{\Phi_\lambda^{(1)}(t - i\omega_1)}{\Phi_\lambda^{(1)}(t)} = \frac{1}{1 - g^{\omega_1} e^{\frac{2\pi t}{\omega_2} + \frac{\pi i \lambda}{\omega_2}}}, \tag{2.4c}$$

$$\frac{\Phi_\lambda^{(1)}(t - i\omega_2)}{\Phi_\lambda^{(1)}(t)} = \frac{1}{1 - g^{\omega_2} e^{\frac{2\pi t}{\omega_1} + \frac{\pi i \lambda}{\omega_1}}}. \tag{2.4d}$$

In a similar way, the Whittaker vectors $\widehat{\Phi}_\lambda^{(\alpha)}$ with $\alpha = 0, 1$ satisfy the difference equations

$$\frac{\widehat{\Phi}_\lambda^{(0)}(t + i\omega_1)}{\widehat{\Phi}_\lambda^{(0)}(t)} = e^{\frac{2\pi i \lambda}{\omega_2}} \Big\{ 1 + q^{-1} g^{\omega_1} e^{-\frac{2\pi t}{\omega_2} - \frac{\pi i \lambda}{\omega_2}} \Big\}, \tag{2.5a}$$

$$\frac{\widehat{\Phi}_\lambda^{(0)}(t + i\omega_2)}{\widehat{\Phi}_\lambda^{(0)}(t)} = e^{\frac{2\pi i \lambda}{\omega_1}} \Big\{ 1 + \widetilde{q}^{\,-1} g^{\omega_2} e^{-\frac{2\pi t}{\omega_1} - \frac{\pi i \lambda}{\omega_1}} \Big\}, \tag{2.5b}$$

$$\frac{\widehat{\Phi}_\lambda^{(1)}(t + i\omega_1)}{\widehat{\Phi}_\lambda^{(1)}(t)} = \frac{e^{\frac{2\pi i \lambda}{\omega_2}}}{1 + q^{-1} g^{\omega_1} e^{-\frac{2\pi t}{\omega_2}}}, \tag{2.5c}$$

$$\frac{\widehat{\Phi}_\lambda^{(1)}(t + i\omega_2)}{\widehat{\Phi}_\lambda^{(1)}(t)} = \frac{e^{\frac{2\pi i \lambda}{\omega_1}}}{1 + \widetilde{q}^{\,-1} g^{\omega_2} e^{-\frac{2\pi t}{\omega_1}}}. \tag{2.5d}$$

Let $\mathcal{S}(y)$ be the function defined in terms of the double sine $S_2(y)$ according to (A.17).[6]

[6] In the main text we shall write $\mathcal{S}(y)$ instead of $\mathcal{S}(y|\boldsymbol{\omega})$ for brevity. We omit such dependence for any other function of such type.

Proposition 2.1 *The Whittaker vectors satisfying the equations (2.4a – 2.5d) are given by the following formulae:*

$$\Phi_\lambda^{(0)}(t) = \mathcal{S}\Big(-it+\omega_1+\omega_2-\tfrac{i\omega_1\omega_2}{2\pi}\log g\Big), \tag{2.6a}$$

$$\Phi_\lambda^{(1)}(t) = \mathcal{S}^{-1}\Big(-it+\omega_1+\omega_2+\frac{\lambda}{2}-\tfrac{i\omega_1\omega_2}{2\pi}\log g\Big), \tag{2.6b}$$

$$\widehat{\Phi}_\lambda^{(0)}(t) = \mathcal{S}\Big(it+\tfrac{1}{2}(\omega_1+\omega_2)-\frac{\lambda}{2}-\tfrac{i\omega_1\omega_2}{2\pi}\log g\Big)e^{\frac{2\pi\lambda t}{\omega_1\omega_2}}, \tag{2.6c}$$

$$\widehat{\Phi}_\lambda^{(1)}(t) = \mathcal{S}^{-1}\Big(it+\tfrac{1}{2}(\omega_1+\omega_2)-\tfrac{i\omega_1\omega_2}{2\pi}\log g\Big)e^{\frac{2\pi\lambda t}{\omega_1\omega_2}}. \tag{2.6d}$$

In a more general way, one can prove the following formulae for the Whittaker vectors $\Phi_\lambda^{(\alpha)}$, $\widehat{\Phi}_\lambda^{(\alpha)}$ with arbitrary values of α:

$$\Phi_\lambda^{(\alpha)}(t) = \int\limits_{\Gamma_\alpha} c(\zeta)e^{\frac{\pi i(2\alpha-1)\zeta^2}{2\omega_1\omega_2}+\frac{2\pi i\zeta}{\omega_1\omega_2}[t+\frac{i\alpha}{2}(\lambda+\omega_1+\omega_2)+\frac{i}{4}(\omega_1+\omega_2)]}\,d\zeta, \tag{2.7a}$$

$$\widehat{\Phi}_\lambda^{(\alpha)}(t) = e^{\frac{2\pi\lambda t}{\omega_1\omega_2}}\int\limits_{\Gamma_\alpha} c(\zeta)e^{\frac{\pi i(2\alpha-1)\zeta^2}{2\omega_1\omega_2}-\frac{2\pi i\zeta}{\omega_1\omega_2}[t+\frac{i(1-\alpha)}{2}(\lambda+\omega_1+\omega_2)-\frac{i}{4}(\omega_1+\omega_2)]}\,d\zeta, \tag{2.7b}$$

where

$$c(\zeta) \equiv \frac{g^{i\zeta}}{\sqrt{\omega_1\omega_2}}\,S_2^{-1}(-i\zeta) \tag{2.8}$$

and the contour Γ_α is chosen in such a way that it passes above the poles of the integrand and escapes to infinity in the sector where the function $e^{\frac{\pi i\alpha\zeta^2}{\omega_1\omega_2}}$ is decaying on the left and in the sector where $e^{\frac{\pi i(\alpha-1)\zeta^2}{\omega_1\omega_2}}$ is decaying on the right. For $\alpha\neq 0,1$ $\Phi_\lambda^{(\alpha)}$, $\widehat{\Phi}_\lambda^{(\alpha)}$ are entire functions of the variable t; for "degenerated cases" $\alpha=0,1$ the integrals in (2.7a), (2.7b) may be evaluated explicitly using the formulae (A.27) and reduce to (2.6); in these cases both vectors are meromorphic functions of t.

Let us note that the function $c(\zeta)$ may be regarded as the q*-deformed Harish-Chandra function* (this term is justified by its role in the asymptotic formulae for the Whittaker functions, see below).

2.2 Whittaker functions

Now we would like to define the q-deformed Whittaker functions as the matrix elements of Whittaker vectors. As it was mentioned before, the standard integral (1.17) is divergent in this case. To regularize the integral, one should deform the integration contour in an appropriate way. Therefore, by the scalar product below we mean the suitable regularization of (1.17).

Definition 2.1 *Let* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ *. The Whittaker functions* $w_\lambda^{(\boldsymbol{\alpha})}(x)$ *corresponding to representation* $(\pi_\lambda, \widetilde{\pi}_\lambda)$ *of the algebra* $\mathcal{D}_{mod}$ *are the matrix elements*

$$w_\lambda^{(\boldsymbol{\alpha})}(x) = e^{-\frac{\pi(\omega_1+\omega_2)x}{\omega_1\omega_2}} \Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} \Phi_\lambda^{(\alpha_2)}\Big) \tag{2.9}$$

Proposition 2.2 *The Whittaker functions (2.9) satisfy the equations*

$$\begin{aligned} w_\lambda^{(\boldsymbol{\alpha})}(x-i\omega_1) + w_\lambda^{(\boldsymbol{\alpha})}(x+i\omega_1) + q^{\alpha_1-\alpha_2} g^{2\omega_1} e^{\frac{2\pi x}{\omega_2}} w_\lambda^{(\boldsymbol{\alpha})}(x+i(\alpha_1-\alpha_2)\omega_1) = \\ -\Big(q e^{\frac{\pi i\lambda}{\omega_2}} + q^{-1} e^{-\frac{\pi i\lambda}{\omega_2}}\Big) w_\lambda^{(\boldsymbol{\alpha})}(x), \end{aligned} \tag{2.10a}$$

$$\begin{aligned} w_\lambda^{(\boldsymbol{\alpha})}(x-i\omega_2) + w_\lambda^{(\boldsymbol{\alpha})}(x+i\omega_2) + \widetilde{q}^{\alpha_1-\alpha_2} g^{2\omega_2} e^{\frac{2\pi x}{\omega_1}} w_\lambda^{(\boldsymbol{\alpha})}(x+i(\alpha_1-\alpha_2)\omega_2) = \\ -\Big(q e^{\frac{\pi i\lambda}{\omega_1}} + q^{-1} e^{-\frac{\pi i\lambda}{\omega_1}}\Big) w_\lambda^{(\boldsymbol{\alpha})}(x). \end{aligned} \tag{2.10b}$$

Let us check (2.10a) formally; we shall discuss the convergence of the integral in (2.9) a little later. Set

$$\mathcal{F}_\lambda^{(\boldsymbol{\alpha})}(x) = \Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} \Phi_\lambda^{(\alpha_2)}\Big). \tag{2.11}$$

The eigenvalue of the Casimir operator $\pi_\lambda(C_2)$ is

$$C_2 = q e^{\frac{\pi i\lambda}{\omega_2}} + q^{-1} e^{-\frac{\pi i\lambda}{\omega_2}}. \tag{2.12}$$

Therefore,

$$\Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} C_2 \Phi_\lambda^{(\alpha_2)}\Big) = (q e^{\frac{\pi i\lambda}{\omega_2}} + q^{-1} e^{-\frac{\pi i\lambda}{\omega_2}}) \mathcal{F}_\lambda^{(\boldsymbol{\alpha})}(x). \tag{2.13}$$

On the other hand,

$$\begin{gathered} \Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} C_2 \Phi_\lambda^{(\alpha_2)}\Big) = \\ \Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} \Big\{q^{H+1} + q^{-H-1} + (q-q^{-1})^2 FE\Big\} \Phi_\lambda^{(\alpha_2)}\Big) = \\ \Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} (q^{H+1} + q^{-H-1}) \Phi_\lambda^{(\alpha_2)}\Big) - (q-q^{-1})^2 e^{\frac{2\pi x}{\omega_2}} \Big(F\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} E \Phi_\lambda^{(\alpha_2)}\Big). \end{gathered} \tag{2.14}$$

Using the definition of the Whittaker vectors (2.2), (2.3), we obtain

$$\begin{gathered} \Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} C_2 \Phi_\lambda^{(\alpha_2)}\Big) = \Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} (q^{H+1} + q^{-H-1}) \Phi_\lambda^{(\alpha_2)}\Big) - \\ e^{\pi i(\alpha_2-\alpha_1)} g^{2\omega_1} e^{\frac{2\pi x}{\omega_2}} \Big(\widehat{\Phi}_\lambda^{(\alpha_1)}, e^{-\frac{\pi x}{\omega_2}H} q^{(\alpha_2-\alpha_1)H} \Phi_\lambda^{(\alpha_2)}\Big) = \\ \Big\{q e^{-i\omega_1\partial_x} + q^{-1} e^{i\omega_1\partial_x} - e^{\pi i(\alpha_2-\alpha_1)} g^{2\omega_1} e^{\frac{2\pi x}{\omega_2}} e^{i(\alpha_1-\alpha_2)\omega_1\partial_x}\Big\} \mathcal{F}_\lambda^{(\boldsymbol{\alpha})}(x) \end{gathered} \tag{2.15}$$

From (2.13) and (2.15) it follows that the matrix coefficient $\mathcal{F}_\lambda^{(\boldsymbol{\alpha})}$ satisfies to equation

$$\begin{gathered}\Big\{qe^{-i\omega_1\partial_x}+q^{-1}e^{i\omega_1\partial_x}-e^{\pi i(\alpha_2-\alpha_1)}g^{2\omega_1}e^{\frac{2\pi x}{\omega_2}}e^{i(\alpha_1-\alpha_2)\omega_1\partial_x}\Big\}\mathcal{F}_\lambda^{(\boldsymbol{\alpha})}(x)=\\ \Big(qe^{\frac{\pi i\lambda}{\omega_2}}+q^{-1}e^{-\frac{\pi i\lambda}{\omega_2}}\Big)\mathcal{F}_\lambda^{(\boldsymbol{\alpha})}(x)\end{gathered}\tag{2.16}$$

Hence, the function

$$w_\lambda^{(\boldsymbol{\alpha})}(x)=e^{-\frac{\pi(\omega_1+\omega_2)x}{\omega_1\omega_2}}\mathcal{F}_\lambda^{(\boldsymbol{\alpha})}(x)\tag{2.17}$$

satisfies (2.10a). ■

Corollary 2.1 *Let the unitary weight be* $\lambda=-i\gamma-\omega_1-\omega_2$. *The Whittaker functions* $w_{-i\gamma-\omega_1-\omega_2}^{(\boldsymbol{\alpha})}\equiv w_\gamma^{(\boldsymbol{\alpha})}$ *are eigenfunctions of the Hamilton operators*

$$\begin{aligned}\mathcal{H}^{(\alpha_1-\alpha_2)}&=e^{i\omega_1\partial_x}\ +\ e^{-i\omega_1\partial_x}+q^{\alpha_1-\alpha_2}g^{2\omega_1}e^{\frac{2\pi x}{\omega_2}}e^{i(\alpha_1-\alpha_2)\omega_1\partial_x},\\ \widetilde{\mathcal{H}}^{(\alpha_1-\alpha_2)}&=e^{i\omega_2\partial_x}\ +\ e^{-i\omega_2\partial_x}+\widetilde{q}^{\,\alpha_1-\alpha_2}g^{2\omega_2}e^{\frac{2\pi x}{\omega_1}}e^{i(\alpha_1-\alpha_2)\omega_2\partial_x}\end{aligned}\tag{2.18}$$

with eigenvalues $\varepsilon_\gamma=e^{\frac{\pi\gamma}{\omega_2}}+e^{-\frac{\pi\gamma}{\omega_2}}$ *and* $\widetilde{\varepsilon}_\gamma=e^{\frac{\pi\gamma}{\omega_1}}+e^{-\frac{\pi\gamma}{\omega_1}}$, *respectively.*

Operators $\mathcal{H}^{(\alpha)}$, $\widetilde{\mathcal{H}}^{(\alpha)}$ are the two dual Hamiltonians of the q-deformed 2-particle Toda chain. If ω_1, ω_2 are real, both $\mathcal{H}^{(\alpha)}$ and $\widetilde{\mathcal{H}}^{(\alpha)}$ are essentially self-adjoint in the space of smooth functions on the line which decrease faster than $e^{-|\omega x|}$, where $\omega=\max(\omega_1,\omega_2)$. When $\omega_1=\overline{\omega}_2$, 'physical' self-adjoint Hamiltonians are $\mathcal{H}^{(\alpha)}+\widetilde{\mathcal{H}}^{(\alpha)}$ and $i(\mathcal{H}^{(\alpha)}-\widetilde{\mathcal{H}}^{(\alpha)})$.

Using the explicit formulae for the Whittaker vectors $\Phi_\lambda^{(\alpha)},\widehat{\Phi}_\lambda^{(\alpha)}$, we may express Whittaker functions $w_\gamma^{(\boldsymbol{\alpha})}$ in integral form:

$$\begin{gathered}w_\gamma^{(\boldsymbol{\alpha})}(x)=N_\gamma e^{-\frac{\pi(\omega_1+\omega_2)x}{\omega_1\omega_2}}\Big(\widehat{\Phi}^{(\alpha_1)}_{-i\gamma-\omega_1-\omega_2}\,,e^{-\frac{\pi x}{\omega_2}H}\Phi^{(\alpha_2)}_{-i\gamma-\omega_1-\omega_2}\Big)=\\ N_\gamma\,e^{\frac{\pi i\gamma x}{\omega_1\omega_2}}\int e^{\frac{2\pi(\omega_1+\omega_2)t}{\omega_1\omega_2}}\ \overline{\widehat{\Phi}^{(\alpha_1)}_{-i\gamma-\omega_1-\omega_2}(\bar{t}\,)}\ \Phi^{(\alpha_2)}_{-i\gamma-\omega_1-\omega_2}(t+x)dt.\end{gathered}\tag{2.19}$$

where one introduces the normalization factor N_γ for the future convenience as follows:

$$N_\gamma=\tfrac{1}{\omega_1\omega_2}\,e^{-\frac{\pi i}{2}[B_{2,2}(i\gamma)-B_{2,2}(0)]}\tag{2.20}$$

with the polynomial $B_{2,2}(z)$ defined by (A.4).

In particular, substituting in (2.19) the expressions (2.6b), (2.6c), and using (A.21) we get for the Whittaker function $w_\gamma^{(0,1)}\equiv w_\gamma^{(-)}$ the integral representation

$$\begin{gathered}w_\gamma^{(-)}(x)=N_\gamma\,e^{\frac{\pi i\gamma x}{\omega_1\omega_2}}\int\limits_{\mathcal{C}_-}\mathcal{S}^{-1}\Big(it+\tfrac{i}{2}\,\gamma-\tfrac{i\omega_1\omega_2}{2\pi}\log g\Big)\times\\ \mathcal{S}^{-1}\Big(-it-ix-\tfrac{i}{2}\,\gamma+\tfrac{1}{2}(\omega_1{+}\omega_2)-\tfrac{i\omega_1\omega_2}{2\pi}\log g\Big)\,e^{\frac{2\pi i\gamma t}{\omega_1\omega_2}}\ dt,\end{gathered}\tag{2.21}$$

where the contour belongs asymptotically to the sectors

$$
\begin{aligned}
\arg\omega_1 + \frac{\pi}{2} \ &< \ \arg t < \frac{1}{2}(\arg\omega_1 + \arg\omega_2) + \pi, \\
\arg\omega_1 - \frac{\pi}{2} \ &< \ \arg t < \frac{1}{2}(\arg\omega_1 + \arg\omega_2)
\end{aligned}
\tag{2.22}
$$

(at this point one can relax the 'physical' constraints imposed on parameters ω_1, ω_2) and lies between the two sets of poles of the integrand:

$$
\begin{aligned}
t^{(-)}_{n_1,n_2} &= -\tfrac{\gamma}{2} + \tfrac{\omega_1\omega_2}{2\pi} \log g + i(n_1\omega_1 + n_2\omega_2)\,, && n_1, n_2 \geq 0, \\
t^{(-)}_{m_1,m_2}(x) &= -x - \tfrac{\gamma}{2} - \tfrac{\omega_1\omega_2}{2\pi} \log g - i\Big[(m_1{+}\tfrac{1}{2})\omega_1 + (m_2{+}\tfrac{1}{2})\omega_2\Big], && m_1, m_2 \geq 0.
\end{aligned}
$$

(See (A.12), (A.17) for the description of the poles and the zeros of $\mathcal{S}(y)$.) The choice of the integration contour assures convergence and provides a natural regularization of the divergent inner product. Indeed, to see that the integral in (2.21) is well defined observe that due to (A.24) the integrand has the asymptotics

$$
e^{-\frac{\pi i t^2}{\omega_1\omega_2} + t(\ldots)}\,.
$$

But in sectors (2.22) the quadratic exponential decreases. Hence, the integral (2.21) is absolutely convergent.

In a similar way, the function $w^{(1,0)}_\gamma \equiv w^{(+)}_\gamma$ corresponding to (2.6a), (2.6d) admits the integral representation

$$
\begin{aligned}
w^{(+)}_\gamma(x) = N_\gamma\, e^{\frac{\pi i \gamma x}{\omega_1\omega_2}} \int\limits_{\mathcal{C}_+} &\mathcal{S}\Big(it + \tfrac{1}{2}(\omega_1{+}\omega_2) - \tfrac{i\omega_1\omega_2}{2\pi}\log g\Big) \times \\
&\mathcal{S}\Big(-it - ix + \omega_1{+}\omega_2 - \tfrac{i\omega_1\omega_2}{2\pi}\log g\Big)\, e^{\frac{2\pi i\gamma t}{\omega_1\omega_2}}\ dt,
\end{aligned}
\tag{2.23}
$$

where the contour belongs asymptotically to the sectors

$$
\begin{aligned}
\frac{1}{2}(\arg\omega_1 + \arg\omega_2) + \pi < \arg\ t < \arg\omega_2 + \frac{3\pi}{2}, \\
\frac{1}{2}(\arg\omega_1 + \arg\omega_2) < \arg t < \arg\omega_2 + \frac{\pi}{2}
\end{aligned}
\tag{2.24}
$$

and lies between the two sets of poles of the integrand:

$$
\begin{aligned}
t^{(+)}_{n_1,n_2} &= \tfrac{\omega_1\omega_2}{2\pi} \log g - i\Big[(n_1{+}\tfrac{1}{2})\omega_1 + (n_2{+}\tfrac{1}{2})\omega_2\Big], && n_1, n_2 \geq 0, \\
t^{(+)}_{m_1,m_2}(x) &= -x - \tfrac{\omega_1\omega_2}{2\pi} \log g + i(m_1\omega_1 + m_2\omega_2), && m_1, m_2 \geq 0.
\end{aligned}
$$

The integral (2.23) is absolutely convergent.

Quite similarly, one can construct the function $w_\gamma^{(0,0)}(x) \equiv w_\gamma^{(0)}(x)$ using the Whittaker vectors $\widehat{\Phi}_\lambda^{(0)}$ and $\Phi_\lambda^{(0)}$:

$$\begin{aligned} w_\gamma^{(0)}(x) = N_\gamma\, e^{\frac{\pi i\gamma x}{\omega_1\omega_2}} \int\limits_{\mathcal{C}_0} \mathcal{S}^{-1}\Big(it + \tfrac{i}{2}\,\gamma - \tfrac{i\omega_1\omega_2}{2\pi}\log g\Big) \times \\ \mathcal{S}\Big(-it - ix + \omega_1 + \omega_2 - \tfrac{i\omega_1\omega_2}{2\pi}\log g\Big)\, e^{\frac{2\pi i\gamma t}{\omega_1\omega_2}}\, dt\,, \end{aligned} \tag{2.25}$$

where the contour $\mathcal{C}_0$ belongs asymptotically to the sectors

$$\begin{aligned} \arg\omega_1 + \frac{\pi}{2} < \arg\, t < \frac{1}{2}(\arg\omega_1 + \arg\omega_2) + \pi, \\ \frac{1}{2}(\arg\omega_1 + \arg\omega_2) < \arg t < \arg\omega_2 + \frac{\pi}{2} \end{aligned} \tag{2.26}$$

and lies below the poles of the integrand.

Thus, the functions (2.21), (2.23), and (2.25) are the proper solutions to corresponding spectral problems

$$\Big[1 + q^{-1}g^{2\omega_1}e^{\frac{2\pi x}{\omega_2}}\Big]w_\gamma^{(-)}(x - i\omega_1) + w_\gamma^{(-)}(x + i\omega_1) = \Big(e^{\frac{\pi\gamma}{\omega_2}} + e^{-\frac{\pi\gamma}{\omega_2}}\Big)w_\gamma^{(-)}(x)\,, \tag{2.27}$$

$$w_\gamma^{(+)}(x - i\omega_1) + \Big[1 + qg^{2\omega_1}e^{\frac{2\pi x}{\omega_2}}\Big]w_\gamma^{(+)}(x + i\omega_1) = \Big(e^{\frac{\pi\gamma}{\omega_2}} + e^{-\frac{\pi\gamma}{\omega_2}}\Big)w_\gamma^{(+)}(x)\,, \tag{2.28}$$

$$w_\gamma^{(0)}(x - i\omega_1) + w_\gamma^{(0)}(x + i\omega_1) + g^{2\omega_1}e^{\frac{2\pi x}{\omega_2}}\,w_\gamma^{(0)}(x) = \Big(e^{\frac{\pi\gamma}{\omega_2}} + e^{-\frac{\pi\gamma}{\omega_2}}\Big)w_\gamma^{(0)}(x)\,. \tag{2.29}$$

Besides, these solutions are the eigenfunctions for the dual spectral problems where $\omega_1 \leftrightarrow \omega_2$.

The solutions $w_\gamma^{(\pm)}$ described above appear to be close to the q-Macdonald functions of the first and second kind which arise in the context of relativistic Toda chain [25]. However, the deformations of the Macdonald function have been investigated in the framework of the standard q-analysis [26] for the typical region $|q| < 1$ (which evidently fails in the case $|q| = 1$) and without any reference to the dual symmetry.

Formulae (2.19) will be referred to as the Gauss-Euler representation for Whittaker functions. The integral representations for $w_\gamma^{(\epsilon)}(x)$, $(\epsilon = 0, \pm 1)$ are the degenerations of more general q-hypergeometric function [27].

We shall see later that the technique of QISM yields a different integral representation for Whittaker functions which is a q-deformation of the Mellin-Barnes integrals.

2.3 Analytic properties

Let us give the summary of the analytic properties of the Whittaker functions which may be derived directly from the Gauss-Euler representation.

Lemma 2.1 *$w_\gamma^{(\pm)}$ and $w_\gamma^{(0)}$ can be extended to the entire functions in $\gamma\in\mathbb{C}$. As a function of $x\in\mathbb{C}$, $w_\gamma^{(-)}(x)$ has poles at*

$$x=-\tfrac{\omega_1\omega_2}{\pi}\,\log g-i(k_1+\tfrac{1}{2})\omega_1-i(k_2+\tfrac{1}{2})\omega_2,\qquad k_1,k_2\geq 0. \tag{2.30}$$

Similarly, the function $w_\gamma^{(+)}(x)$ has poles at

$$x=-\tfrac{\omega_1\omega_2}{\pi}\,\log g+i(k_1+\tfrac{1}{2})\omega_1+i(k_2+\tfrac{1}{2})\omega_2,\qquad k_1,k_2\geq 0. \tag{2.31}$$

The function $w_\gamma^{(0)}(x)$ is an entire one in $x\in\mathbb{C}$.

Lemma 2.2

$$w_\gamma^{(+)}(x)=\mathcal{S}\Big(-ix+\tfrac{1}{2}(\omega_1+\omega_2)-\tfrac{i\omega_1\omega_2}{\pi}\,\log g\Big)\,w_\gamma^{(-)}(x). \tag{2.32}$$

Lemma 2.3 *For any $\gamma\in\mathbb{C}$ such that*

$$\arg\gamma\notin[\arg\omega_2-\tfrac{\pi}{2},\arg\omega_1-\tfrac{\pi}{2}]\bigcup[\arg\omega_2+\tfrac{\pi}{2},\arg\omega_1+\tfrac{\pi}{2}] \tag{2.33}$$

the following asymptotics holds as x tends to infinity in the sector

$$\arg\omega_1+\frac{\pi}{2}<\arg x<\arg\omega_2+\frac{3\pi}{2}: \tag{2.34}$$

$$w_\gamma^{(\epsilon)}(x)=c(\gamma)\,e^{\frac{\pi i\gamma x}{\omega_1\omega_2}}\Big[1+o(1)\Big]+c(-\gamma)\,e^{-\frac{\pi i\gamma x}{\omega_1\omega_2}}\Big[1+o(1)\Big]. \tag{2.35}$$

($\epsilon=0,\pm$), *where the function $c(\gamma)$ is defined by (2.8).*

We shall call $c(\gamma)$ the quantum Harish-Chandra function associated with $U_q(\mathfrak{sl}(2,\mathbb{R}))$.

2.4 Mellin-Barnes representation

To make a comparison with the formulae provided by the Quantum Inverse Scattering Method we need a different integral representation of the Whittaker functions. Put

$$\psi_\gamma^{(\epsilon)}(x)=e^{\frac{\pi i\gamma x}{\omega_1\omega_2}}\int\limits_{\mathcal{C}'_\epsilon}c(\zeta)c(\zeta+\gamma)\,e^{-\frac{\pi i\epsilon}{\omega_1\omega_2}[\zeta^2+\gamma\zeta]}\;e^{\frac{2\pi i\zeta x}{\omega_1\omega_2}}\,d\zeta \tag{2.36}$$

where the contour $\mathcal{C}'_\epsilon$ is above the poles of the integrand and belongs in the left (right) half-plane in $\zeta\in\mathbb{C}$ to the sectors where the exponential $e^{-\frac{\pi(\epsilon-1)\zeta^2}{\omega_1\omega_2}}$ ($e^{-\frac{\pi(\epsilon+1)\zeta^2}{\omega_1\omega_2}}$) is quadratically vanish. The integral (2.36) is absolutely convergent for any $x\in\mathbb{C}$ provided $\epsilon\neq\pm1$. In degenerate case $\epsilon=-1$ the integral is convergent provided the condition

$$\arg x\notin\Big[\arg\omega_2-\frac{\pi}{2},\arg\omega_1-\frac{\pi}{2}\Big], \tag{2.37}$$

while for $\epsilon = 1$ it is defined in the region

$$\arg x \notin \left[\arg \omega_2 + \frac{\pi}{2}, \arg \omega_1 + \frac{\pi}{2}\right]. \tag{2.38}$$

Using the properties of double sine it can be directly verified that the function $\psi_\gamma^{(\epsilon)}(x)$ satisfies to equations (2.10) where $\alpha_1 - \alpha_2 = \epsilon$.

Proposition 2.3 *For* $\epsilon = 0, \pm 1$

$$w_\gamma^{(\epsilon)}(x) = \psi_\gamma^{(\epsilon)}(x). \tag{2.39}$$

The expression (2.36) will be referred to as the (q-deformed) Mellin-Barnes representation for Whittaker functions. It will be shown below that this is the very representation which can be easily generalized to those for N-particle q-deformed Toda chain.

2.5 Limit to $SL(2,\mathbb{R})$ Toda chain

Let $\omega_k > 0$, $(k = 1, 2)$. Suppose that the "coupling constant" $g(\boldsymbol{\omega})$ has the asymptotics such that

$$g^{\omega_1}(\boldsymbol{\omega}) = \frac{2\pi}{\omega_2}\,[1 + O(\omega_2^{-1})\,] \qquad\qquad (\omega_2 \to \infty) \tag{2.40}$$

For example, the simplest (and standard) choice $g^{\omega_1}(\boldsymbol{\omega}) = \frac{q-q^{-1}}{i\omega_1}$ satisfies this condition. After the rescaling $x \to \frac{\omega_2}{\pi}\, x$ the equations (2.27), (2.28), and (2.29) take the form

$$\left\{\left[1 + q^{-1} g^{2\omega_1} e^{2x}\right] e^{-\frac{\pi i \omega_1}{\omega_2}\partial_x} + e^{\frac{\pi i \omega_1}{\omega_2}\partial_x}\right\} w_\gamma^{(-)}(x) = \left(e^{\frac{\pi\gamma}{\omega_2}} + e^{-\frac{\pi\gamma}{\omega_2}}\right) w_\gamma^{(-)}(x)\,, \tag{2.41a}$$

$$\left\{e^{-\frac{\pi i \omega_1}{\omega_2}\partial_x} + \left[1 + q^{-1} g^{2\omega_1} e^{2x}\right] e^{\frac{\pi i \omega_1}{\omega_2}\partial_x}\right\} w_\gamma^{(+)}(x) = \left(e^{\frac{\pi\gamma}{\omega_2}} + e^{-\frac{\pi\gamma}{\omega_2}}\right) w_\gamma^{(+)}(x)\,, \tag{2.41b}$$

$$\left\{e^{-\frac{\pi i \omega_1}{\omega_2}\partial_x} + e^{-\frac{\pi i \omega_1}{\omega_2}\partial_x} + g^{2\omega_1} e^{2x}\right\} w_\gamma^{(0)}(x) = \left(e^{\frac{\pi\gamma}{\omega_2}} + e^{-\frac{\pi\gamma}{\omega_2}}\right) w_\gamma^{(0)}(x)\,, \tag{2.41c}$$

In the limit (2.40) the equations (2.41) are reduced to $SL(2,\mathbb{R})$ Toda equation

$$\left\{p^2 + 4e^{2x}\right\} w_\gamma(x) = \gamma^2 w_\gamma(x)\,. \tag{2.42}$$

where $p = -i\omega_1\partial_x$ and ω_1 plays the role of Planck constant. Note that the more general equation (2.10a) has the same limit (2.42). The solution to (2.42) with appropriate asymptotic behavior is written in terms of Macdonald function

$$w_\gamma(x) = K_{\frac{\gamma}{i\omega_1}}(\tfrac{2}{\omega_1}\, e^x)\,. \tag{2.43}$$

Lemma 2.4

$$\lim_{\omega_2\to\infty} \psi^{(\epsilon)}(\tfrac{\omega_2}{\pi}\,x) \;=\; \frac{1}{\pi\omega_1}\, K_{\frac{\gamma}{i\omega_1}}(\tfrac{2}{\omega_1}\,e^{x})\,. \tag{2.44}$$

Proof. Using the formula

$$\lim_{\omega_2\to\infty} \sqrt{2\pi}\left(\frac{2\pi\omega_1}{\omega_2}\right)^{\frac{1}{2}-\frac{z}{\omega_1}} S_2^{-1}(z) \;=\; \Gamma\Big(\frac{z}{\omega_1}\Big)\,, \tag{2.45}$$

proved in [28], one easily finds that the quantum Harish-Chandra function (2.8) reduces to the usual Γ-function:

$$\lim_{\omega_2\to\infty} c(\zeta) = \tfrac{1}{2\pi\omega_1}\,\omega_1^{\frac{\zeta}{i\omega_1}}\,\Gamma\Big(\frac{\zeta}{i\omega_1}\Big)\,, \tag{2.46}$$

provided that asymptotics (2.40) holds. (This function is closely related to the standard Harish-Chandra function for the Toda chain [6]; the difference with the usual definition is due to a different normalization of solutions). Hence, in the limit $\omega_2 \to \infty$ the rescaled function (2.36) acquires the form

$$\begin{aligned} &\lim_{\omega_2\to\infty} \psi^{(\epsilon)}(\tfrac{\omega_2}{\pi}\,x) \;= \\ &= \frac{1}{\pi\omega_1}\left\{\frac{1}{4\pi\omega_1}\Big(\frac{e^{x}}{\omega_1}\Big)^{\frac{i\gamma}{\omega_1}}\int \Gamma\Big(\frac{\zeta}{i\omega_1}\Big)\Gamma\Big(\frac{\zeta+\gamma}{i\omega_1}\Big)\Big(\frac{e^{x}}{\omega_1}\Big)^{\frac{2i\zeta}{\omega_1}} d\zeta\right\}\,, \end{aligned} \tag{2.47}$$

where the contour is parallel to real axis and above the poles of the integrand. The expression in brackets is exactly the Macdonald function (2.43) in Mellin-Barnes representation. ■

3 N-particle q-Toda chain and duality

The extension of the formalism described above to the case of N-particle Toda chain may be performed directly with the help of the 'free field representation' for $U_q(\mathfrak{sl}(N,\mathbb{R}))$, i.e., the homomorphism of $U_q(\mathfrak{sl}(N,\mathbb{R}))$ into an appropriate multidimensional quantum torus. Instead, we shall describe a different approach based on 'lattice Lax representation with spectral parameter'. As usual, Lax representation allows to construct quantum Hamiltonians for a bunch of related systems: periodic Toda chain, open Toda chain, as well as different degenerate systems obtained by removing some of the potential terms from the Hamiltonians. Of course, the choice of the model in question depends on our choice of the quantum R-matrix. The obvious choice is the standard trigonometric 4×4 R-matrix; to get more freedom in the choice of the model we may use *twisted trigonometric R-matrices*. In all cases, there is a natural homomorphism of the corresponding quantum algebra into the tensor product of noncommutative tori; this allows to introduce the corresponding dual system realized by means of the natural representation of the product of modular dual quantum tori in the same Hilbert space. The entire picture of modular duality is thus fully

generalized to the N-particle case. We would like to point out that in R-matrix formalism it is more convenient to work with $U_q(\mathfrak{gl}(N,\mathbb{R}))$ and reduce the final formulae to the case of $U_q(\mathfrak{sl}(N,\mathbb{R}))$ by the standard way.

The main advantage provided by the use of the lattice Lax representation is the possibility to get inductive integral representations for the wave functions in question and generalization of the above construction to the periodic case as it was done in [29].

3.1 The models

q-Toda chain, or relativistic Toda chain (RTC), was introduced by Ruijsenaars [18]. The periodic chain can be described by the Hamiltonian

$$H_1(x_1,p_1;\dots;x_N,p_N) = \sum_{n=1}^{N} \left\{1 + q^{-1} g^{2\omega_1} e^{\frac{2\pi}{\omega_2}(x_n - x_{n+1})}\right\} e^{\omega_1 p_n} \tag{3.1}$$

where x_n, p_n are the canonical coordinates and momenta with standard commutation relations $[x_n,p_m] = i\delta_{nm}$ and the boundary condition $x_{N+1} = x_1$ is imposed. The system has exactly N mutually commuting Hamiltonians (the polynomial functions of the Weyl variables $u_n^{\pm 1} = e^{\pm\frac{2\pi x_n}{\omega_2}}$, $v_n = e^{\omega_1 p_n}$) [7] .

Guided by the notion of the modular double considered above, one can define the *dual* system which is determined by the Hamiltonian

$$\widetilde{H}_1(x_1,p_1;\dots;x_N,p_N) = \sum_{n=1}^{N} \left\{1 + q^{-1} g^{2\omega_2} e^{\frac{2\pi}{\omega_1}(x_n - x_{n+1})}\right\} e^{\omega_2 p_n} \tag{3.2}$$

with the same boundary condition. It is evident that the systems mutually commute.

Analogously, the open relativistic Toda chain and its dual system are defined by the Hamiltonians

$$h_1(x_1,p_1;\dots;x_N,p_N) = \sum_{n=1}^{N} \left\{1 + q^{-1} g^{2\omega_1} e^{\frac{2\pi}{\omega_2}(x_n - x_{n+1})}\right\} e^{\omega_1 p_n} \tag{3.3}$$

and

$$\widetilde{h}_1(x_1,p_1;\dots;x_N,p_N) = \sum_{n=1}^{N} \left\{1 + q^{-1} g^{2\omega_2} e^{\frac{2\pi}{\omega_1}(x_n - x_{n+1})}\right\} e^{\omega_1 p_n} \tag{3.4}$$

respectively with the boundary condition $x_{N+1} \equiv \infty$. Similarly to the periodic case, each open system possesses exactly N mutually commuting Hamiltonians. Moreover, the Hamiltonians of the dual system commute with those of original one.

The basic goal of the present section is to construct the explicit integral representation of the common eigenfunctions for all Hamiltonians in the case of the *open* N-particle RTC. This will be done in the framework of the QISM approach for the *periodic* RTC.

[7] Higher Hamiltonians will be described below using the standard Lax formalism.

3.2 Twisted trigonometric R-matrix

In order to investigate the relativistic Toda chain using the quantum version of the corresponding classical Lax matrix [30], one needs to introduce the notion of the twisted R-matrix [31]. Let

$$R(z/w) \;=\; \frac{1}{z^2-w^2} \times$$
$$\times \begin{pmatrix} qz^2-q^{-1}w^2 & 0 & 0 & 0 \\ 0 & z^2-w^2 & (q-q^{-1})zw & 0 \\ 0 & (q-q^{-1})zw & z^2-w^2 & 0 \\ 0 & 0 & 0 & qz^2-q^{-1}w^2 \end{pmatrix} \tag{3.5}$$

be the R-matrix in principal gradation satisfying the standard Yang-Baxter equation. Consider the twisting of R-matrix (3.5):

$$R_\theta(z/w) \;=\; F_{21}(\theta)R(z/w)F_{12}^{-1}(\theta) \tag{3.6}$$

with

$$F_{12}(\theta) \;\equiv\; F_{21}^{-1}(\theta) \;=\; \exp\left\{\frac{\theta}{4}\Big(1\otimes\sigma_3 - \sigma_3\otimes 1\Big)\right\} \tag{3.7}$$

where σ_3 is the Pauli matrix. One gets

$$R_\theta(z/w) \;=\; \frac{1}{z^2-w^2}\begin{pmatrix} a(z,w) & 0 & 0 & 0 \\ 0 & b(z,w) & c(z,w) & 0 \\ 0 & c(z,w) & \overline{b}(z,w) & 0 \\ 0 & 0 & 0 & a(z,w) \end{pmatrix} \tag{3.8}$$

where

$$\begin{aligned} a(z,w) \;&=\; qz^2 - q^{-1}w^2 \\ b(z,w) \;&=\; e^{\theta}(z^2-w^2) \\ \overline{b}(z,w) \;&=\; e^{-\theta}(z^2-w^2) \\ c(z,w) \;&=\; (q-q^{-1})zw \end{aligned} \tag{3.9}$$

It is easy to verify that $R_\theta(z/w)$ satisfies the same Yang-Baxter equation as $R(z/w)$.

A quantum Lax operator $L(z)$ is, by definition, a 2×2-matrix

$$L(z) \;=\; \begin{pmatrix} L^{11}(z) & L^{12}(z) \\ L^{21}(z) & L^{22}(z) \end{pmatrix} \tag{3.10}$$

with operator-valued entries which satisfies the fundamental commutation relations

$$R_\theta(z/w)L(z)\otimes L(w) \;=\; (1\otimes L(w))(L(z)\otimes 1)R_\theta(z/w)\,. \tag{3.11}$$

We define the quantum determinant of the matrix (3.10) by the formula

$$\det\nolimits_q L(z) = L^{11}(zq^{1/2})L^{22}(zq^{-1/2}) - e^{\theta}L^{12}(zq^{1/2})L^{21}(zq^{-1/2})\,. \tag{3.12}$$

3.3 Lax operator and monodromy matrix

As usual in the Quantum Inverse Scattering Method, the entries of the quantum Lax operator generate the basic Hopf algebra $\mathcal{A}_R$ (defined implicitly by the fundamental commutation relation (3.11) which underlies all the associated quantum integrable systems; to get a particular system, we need to fix its representation. The representation which yields the q-deformed Toda chain is provided by the following construction.

Let $\omega_1, \omega_2 \in \mathbb{C}$. We consider a lattice system with local quantum Lax operators

$$L_n(z) = \begin{pmatrix} z - z^{-1} e^{\omega_1 p_n} & g^{\omega_1}\, e^{-\frac{2\pi x_n}{\omega_2}} \\ -g^{\omega_1}\, e^{\frac{2\pi x_n}{\omega_2} + \omega_1 p_n} & 0 \end{pmatrix} \tag{3.13}$$

where x_n, p_n are the canonical coordinates and momenta with the commutation relations $[x_n, p_m] = i\delta_{nm}$ and g is a real parameter (possibly depending on $\boldsymbol{\omega}$). On the classical level the Lax matrices (3.13) have been introduced in [30].

Proposition 3.1 *Lax operator (3.13) satisfies the commutation relations (3.11) with the quantum R-matrix (3.8), (3.9), where*

$$q = e^{i\pi \frac{\omega_1}{\omega_2}}\,, \tag{3.14}$$

and

$$e^{\theta} = q. \tag{3.15}$$

The monodromy matrix for the N-*periodic* chain is defined in the standard way:

$$T_{N}(z) = L_{N}(z) \ldots L_1(z) \equiv \begin{pmatrix} A_{N}(z) & B_{N}(z) \\ C_{N}(z) & D_{N}(z) \end{pmatrix}. \tag{3.16}$$

By the usual Hopf algebra properties, the entries of $T(z)$ satisfy the same commutation relations as the corresponding entries of the Lax operators.

The quantum determinant of the Lax operator (3.13) is

$$\mathrm{det}_q L(z) = g^{2\omega_1} e^{\omega_1 p_n}\,. \tag{3.17}$$

It is simple to show that the quantum determinant of monodromy matrix

$$\mathrm{det}_q T_{N}(z) = A_{N}(zq^{1/2}) D_{N}(zq^{-1/2}) - q B_{N}(zq^{1/2}) C_{N}(zq^{-1/2}) \tag{3.18}$$

obeys the property

$$\mathrm{det}_q T_{N}(z) = \mathrm{det}_q L_{N}(z) \cdot \ldots \cdot \mathrm{det}_q L_1(z)\,. \tag{3.19}$$

Hence, due to (3.17),

$$\mathrm{det}_q T_{N}(z) = g^{2N\omega_1} \prod_{n=1}^{N} e^{\omega_1 p_n}\,. \tag{3.20}$$

Note that in the twisted case the quantum determinant is no longer a central element in the quantum algebra.

Following the same line of argument as in section 1, we may introduce the modular dual system by

$$\widetilde{L}_n(z) \;=\; \left(\begin{array}{cc} z - z^{-1} e^{\omega_2 p_n} & g^{\omega_2}\, e^{-\frac{2\pi x_n}{\omega_1}} \\ -g^{\omega_2}\, e^{\frac{2\pi x_n}{\omega_1}+\omega_2 p_n} & 0 \end{array} \right) . \tag{3.21}$$

The operator (3.21) satisfies the commutation relation (3.11) with the twisted R-matrix (3.8), (3.9) with the only change $q \to \widetilde{q}\,,\ \theta \to \widetilde{\theta}$ where $\widetilde{q} = e^{\widetilde{\theta}} = e^{i\pi \frac{\omega_2}{\omega_1}}$. The dual monodromy matrix is defined by

$$\widetilde{T}_N(z) = \widetilde{L}_N(z) \ldots \widetilde{L}_1(z) \equiv \left(\begin{array}{cc} \widetilde{A}_N(z) & \widetilde{B}_N(z) \\ \widetilde{C}_N(z) & \widetilde{D}_N(z) \end{array} \right). \tag{3.22}$$

The system describing by the Lax operators (3.13), (3.21) may be referred to as the modular relativistic Toda chain.

3.4 Hamiltonians

As usual, the transfer matrix

$$t_N(z) = A_N(z) + D_N(z) \tag{3.23}$$

satisfies to commutation relations

$$[t_N(z), t_N(w)] = 0\,. \tag{3.24}$$

The same is true for the dual transfer matrix

$$\widetilde{t}_N(z) = \widetilde{A}_N(z) + \widetilde{D}_N(z); \tag{3.25}$$

moreover, the modular duality implies that

$$[t_N(z), \widetilde{t}_N(w)] = 0. \tag{3.26}$$

Clearly, $t_N(z)$ has the following structure:

$$t_N(z) = \sum_{k=0}^{N} (-1)^k z^{N-2k} H_k(x_1, p_1; \ldots ; x_N, p_N) \tag{3.27}$$

where $H_0 = 1$ and

$$H_N(p_1, \ldots , p_N) = \exp \left\{ \omega_1 \sum_{n=1}^{N} p_n \right\} , \tag{3.28}$$

$$H_1(x_1,p_1;\dots;x_N,p_N) = \sum_{n=1}^{N} \left\{1 + q^{-1} g^{2\omega_1} e^{\frac{2\pi}{\omega_2}(x_n - x_{n+1})}\right\} e^{\omega_1 p_n}\,, \tag{3.29}$$

$$H_{N-1}(x_1,p_1;\dots;x_N,p_N) = H_N \sum_{n=1}^{N} \left\{1 + q^{-1} g^{2\omega_1} e^{\frac{2\pi}{\omega_2}(x_{n-1} - x_n)}\right\} e^{-\omega_1 p_n}\,, \tag{3.30}$$

where in (3.29), (3.30) the periodicity is assumed: $x_{N+1} = x_1$. Hence, due to (3.24) the *periodic* RTC has exactly N commuting operators.

The following statement is true: the operator $A_N(z)$ is the generating function for the Hamiltonians of N-particle *open* RTC:

$$A_N(z) = \sum_{k=0}^{N} (-1)^k z^{N-2k} h_k(x_1,p_1;\dots;x_N,p_N) \tag{3.31}$$

where $h_0 = 1$ and

$$h_N(p_1,\dots,p_N) = \exp\left\{\omega_1 \sum_{n=1}^{N} p_n\right\}\,, \tag{3.32}$$

$$h_1(x_1,p_1;\dots;x_N,p_N) = \sum_{n=1}^{N} \left\{1 + q^{-1} g^{2\omega_1} e^{\frac{2\pi}{\omega_2}(x_n - x_{n+1})}\right\} e^{\omega_1 p_n}\,, \tag{3.33}$$

$$h_{N-1}(x_1,p_1;\dots;x_N,p_N) = h_N \sum_{n=1}^{N} \left\{1 + q^{-1} g^{2\omega_1} e^{\frac{2\pi}{\omega_2}(x_{n-1} - x_n})\right\} e^{-\omega_1 p_n} \tag{3.34}$$

assuming $x_{N+1} \equiv \infty$ in (3.33) and $x_0 \equiv -\infty$ in (3.34).

The second set of the Hamiltonians $\widetilde{H}_1,\dots\widetilde{H}_N$ and $\widetilde{h}_1,\dots\widetilde{h}_N$ are obtained from the former one by the flip $\omega_1 \leftrightarrow \omega_2$.

Lemma 3.1 *1. Suppose that ω_1,ω_2 are real; then all coefficients of $t_N(z)$, $\widetilde{t}_N(z)$ and $A_N(z), \widetilde{A}_N(z)$ are formally self-adjoint in $L_2(\mathbb{R}^N)$.*

2. Suppose that $\operatorname{Im}\omega_1 \neq 0$ and $\overline{\omega}_1 = \omega_2$; then all coefficients of $t_N(z)$, $\widetilde{t}_N(z)$ and $A_N(z)$, $\widetilde{A}_N(z)$ are normal operators and their 'real' and 'imaginary' parts ($X + \widetilde{X}$, $i(X - \widetilde{X})$) are formally self-adjoint.

3.5 Integral representation for the wave functions: inductive procedure

Our goal is to get an inductive integral representation for the wave functions of the multi-particle open relativistic Toda chain. The approach described below is an analytical version of the algebraic method of separation of variables invented by Sklyanin [32].

Set $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_{N-1})\in\mathbb{R}^{N-1}$, $\boldsymbol{x}=(x_1,\ldots,x_{N-1})\in\mathbb{R}^{N-1}$. Let $\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})$ be the common wave function for the dual open RTC systems with $N-1$ particles:

$$A_{N-1}(z)\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})=\prod_{m=1}^{N-1}\left(z-z^{-1}e^{\frac{2\pi\gamma_m}{\omega_2}}\right)\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})\,, \tag{3.35}$$

$$\widetilde{A}_{N-1}(z)\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})=\prod_{m=1}^{N-1}\left(z-z^{-1}e^{\frac{2\pi\gamma_m}{\omega_1}}\right)\psi_{\boldsymbol{\gamma}}(\boldsymbol{x}). \tag{3.36}$$

The key point of the inductive procedure (described for the first time in [9] for the ordinary Toda chain) is to compute the action on $\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})$ of the N-particle Hamiltonians. It turns out that such an action 'preserves' the form of wave function [8]. This computation, which starts with the case $N=2$, is based on the ordinary RTT commutation relations for the quantum monodromy matrix; the inductive formula given below is based on a self-consistent choice of the normalization for the wave functions.

Proposition 3.2 *There exists the unique solution $\psi_{\gamma_1,\ldots,\gamma_{N-1}}(x_1,\ldots,x_{N-1})$ to the common spectral problem (3.35), (3.36) such that for any $N\geq 2$ the eigenfunction $\psi_{\boldsymbol{\gamma}}$ is an entire function in $\boldsymbol{\gamma}\in\mathbb{C}^{N-1}$ satisfying to relations*

$$A_N\Big(e^{\frac{2\pi\gamma_j}{\omega_2}}\Big)\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})=q^{-1}(-ig^{\omega_1})^N\exp\left\{\frac{2\pi}{\omega_2}\sum_{m=1}^{N-1}\gamma_m\right\}\,e^{-\frac{2\pi x_N}{\omega_2}}\psi_{\boldsymbol{\gamma}-i\omega_1\boldsymbol{e}_j}(\boldsymbol{x}), \tag{3.37}$$

$$\widetilde{A}_N\Big(e^{\frac{2\pi\gamma_j}{\omega_1}}\Big)\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})=\widetilde{q}^{-1}(-ig^{\omega_2})^N\exp\left\{\frac{2\pi}{\omega_1}\sum_{m=1}^{N-1}\gamma_m\right\}\,e^{-\frac{2\pi x_N}{\omega_1}}\psi_{\boldsymbol{\gamma}-i\omega_2\boldsymbol{e}_j}(\boldsymbol{x}), \tag{3.38}$$

where $\{\boldsymbol{e}_j\}$ be the standard basis of $\mathbb{R}^{N-1}$.

As a comment to the proposition above we remark that $C_N(z)=-q^{-1}g^{\omega_1}e^{\frac{2\pi x_N}{\omega_2}}e^{\omega_1 p_N}A_{N-1}(z)$. Hence, the compatibility of (3.35) and (3.37) follows from the quadratic RTT relations; the argument for the dual system is completely similar.

Assuming that such a function is known, the heuristic idea behind the inductive integral representation for the N-particle wave function $\psi_{\lambda_1,\ldots,\lambda_N}(x_1,\ldots,x_N)$ is to represent it as the generalized Fourier transform with respect to the $N-1$-particle wave function $\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})$. Note that the "one-particle" solution is $\psi_{\gamma_1}(x_1)=e^{\frac{2\pi i\gamma_1 x_1}{\omega_1\omega_2}}$; it is easy to verify directly that this trivial function satisfies the conditions of proposition 3.2, which forms the induction basis. The exact statement is given by theorem 3.1 below; here we represent the essential ideas how to arrive to this statement.

[8] This idea goes back to M. Gutzviller [33] who explicitly calculated such an action on the 2 and 3-particle eigenfunctions for the Toda chain.

Introduce an auxiliary wave function by

$$\Psi_{\boldsymbol{\gamma},\varepsilon}(x_1,\ldots,x_N) \stackrel{\text{def}}{=} \exp\left\{\frac{\pi i}{\omega_1\omega_2}\left[\Big(\sum_{m=1}^{N-1}\gamma_m\Big)^2 - \varepsilon\sum_{m=1}^{N-1}\gamma_m\right]\right\} e^{\frac{2\pi i}{\omega_1\omega_2}\left(\varepsilon-\sum\limits_{m=1}^{N-1}\gamma_m\right)x_N}\,\psi_{\boldsymbol{\gamma}}(x_1,\ldots,x_{N-1})\,. \tag{3.39}$$

Generalizing proposition 3.2, one can prove the following result.

Proposition 3.3 *The action of $A_N(z)$, $\widetilde{A}_N(z)$ on the auxiliary wave function (3.39) is given by*

$$\begin{aligned} A_N(z)\Psi_{\boldsymbol{\gamma},\varepsilon} = &\left(z - z^{-1}e^{\frac{2\pi}{\omega_2}\left(\varepsilon-\sum\limits_{m=1}^{N-1}\gamma_m\right)}\right)\prod_{j=1}^{N-1}\left(z - z^{-1}e^{\frac{2\pi\gamma_j}{\omega_2}}\right)\Psi_{\boldsymbol{\gamma},\varepsilon} + \\ &+(-ig^{\omega_1})^N e^{\frac{\pi\varepsilon}{\omega_2}}\sum_{j=1}^{N-1}\Psi_{\boldsymbol{\gamma}-i\omega_1\boldsymbol{e}_j,\varepsilon}\prod_{s\neq j}\frac{z - e^{\frac{2\pi\gamma_s}{\omega_2}}z^{-1}}{e^{\frac{\pi\gamma_j}{\omega_2}} - e^{\frac{2\pi\gamma_s}{\omega_2}}e^{-\frac{\pi\gamma_j}{\omega_2}}}\,, \end{aligned} \tag{3.40}$$

$$\begin{aligned} \widetilde{A}_N(z)\Psi_{\boldsymbol{\gamma},\varepsilon} = &\left(z - z^{-1}e^{\frac{2\pi}{\omega_1}\left(\varepsilon-\sum\limits_{m=1}^{N-1}\gamma_m\right)}\right)\prod_{j=1}^{N-1}\left(z - z^{-1}e^{\frac{2\pi\gamma_j}{\omega_1}}\right)\Psi_{\boldsymbol{\gamma},\varepsilon} + \\ &+(-ig^{\omega_2})^N e^{\frac{\pi\varepsilon}{\omega_1}}\sum_{j=1}^{N-1}\Psi_{\boldsymbol{\gamma}-i\omega_2\boldsymbol{e}_j,\varepsilon}\prod_{s\neq j}\frac{z - e^{\frac{2\pi\gamma_s}{\omega_1}}z^{-1}}{e^{\frac{\pi\gamma_j}{\omega_1}} - e^{\frac{2\pi\gamma_s}{\omega_1}}e^{-\frac{\pi\gamma_j}{\omega_1}}}\,. \end{aligned} \tag{3.41}$$

Let us write formally

$$\psi_{\lambda_1,\ldots,\lambda_N}(x_1,\ldots,x_N) = \int \mu(\boldsymbol{\gamma})\mathcal{Q}(\boldsymbol{\gamma}|\boldsymbol{\lambda})\Psi_{\boldsymbol{\gamma},\lambda_1+\ldots+\lambda_N}\,d\boldsymbol{\gamma}. \tag{3.42}$$

where

$$\mu(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \prod_{j<k}\left\{4\omega_1\omega_2\,\sinh\frac{\pi}{\omega_1}(\gamma_j-\gamma_k)\cdot\sinh\frac{\pi}{\omega_2}(\gamma_j-\gamma_k)\right\}. \tag{3.43}$$

and $\mathcal{Q}(\boldsymbol{\gamma}|\boldsymbol{\lambda})$ is an unknown kernel. The form of the integrand in (3.42) is a natural generalization of the one for the usual Toda chain [9]; the latter case actually corresponds to the limit $\omega_2\to\infty$ of RTC model, and in this limit the function $\mu(\boldsymbol{\gamma})$ reduces to the Sklyanin measure [32].

By assumption, $\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})$ satisfies, in particular, the equations (3.35) and (3.37); on the other hand, if we apply $A_N(z)$ and $A_{N+1}(e^{\frac{\pi\lambda_n}{\omega_2}})$ to (3.42) and demand that similar equations hold for the function $\psi_{\lambda_1,\ldots,\lambda_N}$ (see exact formulae (3.48) and (3.49) below), these requirements yield, after an appropriate deformation of the integration contour, the difference equations for the Fourier amplitude $\mathcal{Q}(\boldsymbol{\gamma}|\boldsymbol{\lambda})$:

$$i^N\mathcal{Q}(\boldsymbol{\gamma}+i\omega_1\boldsymbol{e}_j|\boldsymbol{\lambda}) = \Big(\frac{g^{\omega_1}}{2}\Big)^{-N}\prod_{k=1}^{N}\sinh\frac{\pi}{\omega_2}(\gamma_j-\lambda_k)\mathcal{Q}(\boldsymbol{\gamma}|\boldsymbol{\lambda})\,, \tag{3.44}$$

$$i^{N-1}\mathcal{Q}(\boldsymbol{\gamma}|\boldsymbol{\lambda}-i\omega_1\boldsymbol{e}'_n)=\Big(\frac{g^{\omega_1}}{2}\Big)^{-N+1}\prod_{j=1}^{N-1}\sinh\frac{\pi}{\omega_2}(\gamma_j-\lambda_n)\mathcal{Q}(\boldsymbol{\gamma}|\boldsymbol{\lambda})\,, \tag{3.45}$$

where $\{\boldsymbol{e}'_n\}$ denote the standard basis in $\mathbb{R}^N$. It is clear that equations (3.44), (3.45) can be factorized into the first order equations of Baxter's type. Assume for the definiteness that $\frac{\omega_1}{\omega_2}\notin\mathbb{R}_-$. Up to a quasiconstant, their solutions can be expressed in terms of the double sine $S_2(z)$ according to equations (A.10); to eliminate the quasiconstant (which must be set to 1), we use the dual ($\omega_1\leftrightarrow\omega_2$) counterpart of (3.44), (3.45) (which arise due to similar reasoning from (3.36), (3.38)) as well as the requirement of analyticity with respect to parameters $\lambda_1,\dots,\lambda_N$.

The formal computation sketched above must be matched by a series of estimates justifying the deformation of the integration contours. The induction basis is given by a careful study of the cases, where $N=2,3$. In this way we arrive at the following result:

Theorem 3.1 *Let $\psi_{\gamma_1,\dots,\gamma_{N-1}}(x_1,\dots,x_{N-1})$ be the solution of equations (3.35)-(3.38). Let $c(\gamma)$ be the rank 1 quantum Harish-Chandra function defined by (2.8). Introduce*

$$\mathcal{Q}(\boldsymbol{\gamma}|\boldsymbol{\lambda})=\prod_{j=1}^{N-1}\prod_{n=1}^{N}c(\gamma_j-\lambda_n). \tag{3.46}$$

Let $\psi_{\lambda_1,\dots,\lambda_N}$ be the function defined by the following integral

$$\psi_{\lambda_1,\dots,\lambda_N}(x_1,\dots,x_N)=\int\limits_{\mathcal{C}}\mu(\boldsymbol{\gamma})\mathcal{Q}(\boldsymbol{\gamma}|\boldsymbol{\lambda})\Psi_{\boldsymbol{\gamma},\lambda_1+\dots+\lambda_N}(x_1,\dots,x_N)\,d\boldsymbol{\gamma} \tag{3.47}$$

where the auxiliary function $\Psi_{\boldsymbol{\gamma},\lambda_1+\dots+\lambda_N}(x_1,\dots,x_N)$ is defined by (3.39) and the contour of integration in the multiple integral is chosen in such a way that
(a). $\operatorname{Im}\gamma_j>\max_k\{\operatorname{Im}\lambda_k\}$;
(b). The left end of the contour escapes to infinity in the sectors

$$\frac{1}{2}(\arg\omega_1+\arg\omega_2)+\pi<\arg\gamma_j<\arg\omega_2+\frac{3\pi}{2}\,;$$

(c). The right end of the contour escapes to infinity in the sectors

$$\arg\omega_1-\frac{\pi}{2}<\arg\gamma_j<\arg\omega_2+\frac{\pi}{2}\,.$$

Then the function (3.47) is a common eigenfunction for N-particle open RTC. Namely, it satisfies to the following properties:

(i) $\psi_{\lambda_1,\dots,\lambda_N}$ is an entire function in $\boldsymbol{\lambda}\in\mathbb{C}^N$;

(ii) $\psi_{\lambda_1,\dots,\lambda_N}$ is the solution to the following set of equations:

$$A_N(z)\psi_{\lambda_1,\dots,\lambda_N}=\prod_{k=1}^{N}\Big(z-z^{-1}e^{\frac{2\pi\lambda_k}{\omega_2}}\Big)\,\psi_{\lambda_1,\dots,\lambda_N}\,, \tag{3.48}$$

$$A_{N+1}\Big(e^{\frac{\pi\lambda_n}{\omega_2}}\Big)\psi_{\lambda_1,\ldots,\lambda_N} \; =$$

$$= \; q^{-1}(-ig^{\omega_1})^{N+1}\exp\left\{\frac{2\pi}{\omega_2}\sum_{k=1}^{N}\lambda_k\right\}\; e^{-\frac{2\pi x_{N+1}}{\omega_2}}\psi_{\lambda_1,\ldots,\lambda_n-i\omega_1,\ldots,\lambda_N}\;, \tag{3.49}$$

$$\widetilde{A}_N(z)\psi_{\lambda_1,\ldots,\lambda_N} = \prod_{k=1}^{N}\Big(z - z^{-1}e^{\frac{2\pi\lambda_k}{\omega_1}}\Big)\;\psi_{\lambda_1,\ldots,\lambda_N}\;, \tag{3.50}$$

$$\widetilde{A}_{N+1}\Big(e^{\frac{\pi\lambda_n}{\omega_1}}\Big)\psi_{\lambda_1,\ldots,\lambda_N} \; =$$

$$= \; q^{-1}(-ig^{\omega_2})^{N+1}\exp\left\{\frac{2\pi}{\omega_1}\sum_{k=1}^{N}\lambda_k\right\}\; e^{-\frac{2\pi x_{N+1}}{\omega_1}}\psi_{\lambda_1,\ldots,\lambda_n-i\omega_2,\ldots,\lambda_N}\;. \tag{3.51}$$

By inductive application of the formula (3.47), starting with trivial one-particle wave function $\psi_{\gamma_1}(x_1) = e^{\frac{2\pi i\gamma_1 x_1}{\omega_1\omega_2}}$, we get an explicit solution for the N-particle system.

Theorem 3.2 *Let* $||\gamma_{jk}||_{j,k=1}^{N}$ *be a lower triangular* $N\times N$ *matrix and let the last row* $(\gamma_{N1},\ldots,\gamma_{NN})$ *be identified with* $(\lambda_1,\ldots,\lambda_N)$.

(i) The solution to (3.48)-(3.51) can be written in the form:

$$\psi_{\lambda_1,\ldots,\lambda_N}(x_1,\ldots,x_N) \; =$$

$$\int\limits_{\mathcal{D}_N}\prod_{n=1}^{N-1}\Bigg\{\Big(\prod_{\substack{j,k=1\\ j<k}}^{n}4\omega_1\omega_2\,\sinh\frac{\pi}{\omega_1}(\gamma_{nj}-\gamma_{nk})\cdot\sinh\frac{\pi}{\omega_2}(\gamma_{nj}-\gamma_{nk})\Big)\prod_{j=1}^{n}\prod_{k=1}^{n+1}c(\gamma_{nj}-\gamma_{n+1,k})\times$$
$$\exp\left[\frac{\pi i}{\omega_1\omega_2}\left(\Big(\sum_{m=1}^{N}\gamma_{nm}\Big)^2-\sum_{k,m=1}^{N}\gamma_{n+1,k}\gamma_{nm}\right)\right]\Bigg\}\times \tag{3.52}$$
$$\exp\left[\frac{2\pi i}{\omega_1\omega_2}\sum_{n,m=1}^{N}x_n\Big(\gamma_{nm}-\gamma_{n-1,m}\Big)\right]\prod_{\substack{j,k=1\\ j\geq k}}^{N-1}d\gamma_{jk}$$

where the integral should be understood as follows. We integrate from top to bottom of the lower triangular matrix: first we integrate on γ_{11} *over the line* $\mathrm{Im}\,\gamma_{11} > \max\{\mathrm{Im}\,\gamma_{21},\mathrm{Im}\,\gamma_{22}\}$*; then we integrate over the set* $(\gamma_{21},\gamma_{22})$ *along the lines* $\mathrm{Im}\,\gamma_{2j} > \max_m\{\mathrm{Im}\gamma_{3m}\}$ *and so on. The last integrations should be performed over the variables* $(\gamma_{N-1,1}\ldots,\gamma_{N-1,N-1})$ *along the lines* $\mathrm{Im}\gamma_{N-1,k} > \max_m\{\mathrm{Im}\,\gamma_{N,m}\}$. *The asymptotic behaviour of all contours is chosen in the same way as in the previous theorem.*

(ii) The wave function $\psi_{\boldsymbol{\lambda}}$ *has the following asymptotic behaviour as* $x_j - x_k\to-\infty$ *inside the positive Weyl chamber* $P_+ = \{(x_1,\ldots,x_N); x_1 < x_2 < \ldots < x_N\}$ *:*

$$\psi_{\lambda_1,\ldots,\lambda_N}(x_1,\ldots,x_N)\sim\sum_{s\in W}\mathrm{C}(s\boldsymbol{\lambda})\,\exp\left\{\frac{\pi i}{\omega_1\omega_2}\,(s\boldsymbol{\lambda},\boldsymbol{x})\right\} \tag{3.53}$$

where

$$\mathrm{C}(\boldsymbol{\lambda}) \,=\, \prod_{j<k} e^{-\frac{\pi i}{\omega_1\omega_2}\lambda_j\lambda_k} c(\lambda_j-\lambda_k)\,, \tag{3.54}$$

the sum is over all permutation of $(\lambda_1,\ldots,\lambda_N)$ *and* $(\,.\,,.)$ *is standard scalar product in* $\mathbb{R}^N$ *.*

We will refer to (3.54) as the quantum Harish-Chandra function for the modular double corresponding to $U_q(\mathfrak{gl}(n))$.

Remark 3.1 *In the case of* $N=2$ *the solution (3.52) has the following form*

$$\psi_{\lambda_1,\lambda_2}(x_1,x_2) \,=\, \\ = e^{\frac{2\pi i}{\omega_1\omega_2}(\lambda_1+\lambda_2)x_2} \int\limits_{\mathcal{D}_2} e^{\frac{\pi i}{\omega_1\omega_2}[\gamma_{11}^2-(\lambda_1+\lambda_2)\gamma_{11}]} c(\gamma_{11}-\lambda_1)c(\gamma_{11}-\lambda_2)\, e^{\frac{2\pi i\gamma_{11}}{\omega_1\omega_2}(x_1-x_2)}\, d\gamma_{11}\ . \tag{3.55}$$

Changing the integration variable $\gamma_{11}\to\zeta+\lambda_1$ *and letting* $x=x_1-x_2$ *one can obtain (up to simple* $GL(1)$ *factor) the function* $\psi_{\lambda_1-\lambda_2}(x)$ *which coincides with* $U_q(\mathfrak{sl}(2))$ *solution (2.36) for* $\epsilon=-1$ *.*

Corollary 3.1 *Let the periods* ω_1,ω_2 *be real positive numbers. Fix the following choice of the coupling constant* $g^{\omega_1}=\frac{q-q^{-1}}{i\omega_1}$ *and let* $\boldsymbol{\rho}$ *be the half-sum of positive roots of* $\mathfrak{sl}(N,\mathbb{R})$ *written in standard basis of* $\mathbb{R}^N$ *. After rescaling* $x_k\to\frac{\omega_2}{2\pi}x_k$ *and sending* $\omega_2\to\infty$ *one obtains in this limit* $\psi_{\boldsymbol{\lambda}}(\frac{\omega_2}{2\pi}\boldsymbol{x})\to\psi^{(TC)}_{\boldsymbol{\lambda}}(\boldsymbol{x})$ *, where* $\psi^{(TC)}_{\boldsymbol{\lambda}}(\boldsymbol{x})$ *is a solution of* $GL(N,\mathbb{R})$ *open Toda chain in the Mellin-Barnes form [9]. In terms of the classical* $GL(N,\mathbb{R})$ *Whittaker functions* $W(\boldsymbol{x};\boldsymbol{\lambda})$ *[4] it can be written in the form*

$$\psi^{(TC)}_{\boldsymbol{\lambda}}(\boldsymbol{x}) \,=\, \omega_1^{-2i(\boldsymbol{\lambda},\boldsymbol{\rho})/\omega_1} \prod_{j<k} \pi^{-1/2}\Gamma\Big(\frac{\lambda_j-\lambda_k}{i\omega_1}+\frac{1}{2}\Big)\ \ W(\boldsymbol{x};\boldsymbol{\lambda}), \tag{3.56}$$

In this limit the quantum Harish-Chandra function (3.54) reduces to the standard one:

$$\mathrm{C}(\boldsymbol{\lambda}) \,\to\, \omega_1^{-2i(\boldsymbol{\lambda},\boldsymbol{\rho})/\omega_1} \prod_{j<k}\Gamma\Big(\frac{\lambda_j-\lambda_k}{i\omega_1}\Big). \tag{3.57}$$

Remark 3.2 *To generalize the solution (2.36) to an arbitrary* N *-particle case one should deal with the Lax operators*

$$L_n^{(\epsilon)}(z) \,=\, \left(\begin{array}{cc} z-z^{-1}e^{\omega_1 p_n} & g^{\omega_1}e^{-\frac{2\pi x_n}{\omega_2}+\frac{1+\epsilon}{2}\omega_1 p_n} \\ -g^{\omega_1}e^{\frac{2\pi x_n}{\omega_2}+\frac{1-\epsilon}{2}\omega_1 p_n} & 0 \end{array}\right)\,, \tag{3.58}$$

$$\widetilde{L}_n^{(\epsilon)}(z) \;=\; \begin{pmatrix} z - z^{-1}e^{\omega_2 p_n} & g^{\omega_2} e^{-\frac{2\pi x_n}{\omega_1}+\frac{1+\epsilon}{2}\omega_2 p_n} \\ -g^{\omega_2} e^{\frac{2\pi x_n}{\omega_1}+\frac{1-\epsilon}{2}\omega_2 p_n} & 0 \end{pmatrix} \tag{3.59}$$

which satisfy the RTT relation with the same twisted R-matrix. Applying the method described above to this model, one obtains the solution of the same structure as (3.52) with the additional factor $-\epsilon$ in front of quadratic form in γ-variables. Note that different deformations of the relativistic Toda chain has been introduced in [34] on purely algebraic level.

3.6 Periodic q-Toda chain

In this section we will formulate briefly the extension of our approach to the case of periodic modular q-Toda chain. The details will be published in a separate publication. In the same way as it was done in the case of the open chain, we can calculate the action of the transfer matrices $t_N(z)$ and $\widetilde{t}_N(z)$ on the auxiliary wave function (3.39). One can prove the following result.

Proposition 3.4 *The action of $t_N(z)$, $\widetilde{t}_N(z)$ on the auxiliary wave function (3.39) is given by*

$$\begin{gathered} t_N(z)\Psi_{\boldsymbol{\gamma},\varepsilon} = \left(z - z^{-1} e^{\frac{2\pi}{\omega_2}\left(\varepsilon - \sum\limits_{m=1}^{N-1}\gamma_m\right)}\right) \prod_{j=1}^{N-1}\left(z - z^{-1} e^{\frac{2\pi\gamma_j}{\omega_2}}\right)\Psi_{\boldsymbol{\gamma},\varepsilon} + \\ e^{\frac{\pi\varepsilon}{\omega_2}} \sum_{j=1}^{N-1}\left\{(-ig^{\omega_1})^N \Psi_{\boldsymbol{\gamma}-i\omega_1\boldsymbol{e}_j,\varepsilon} + (ig^{\omega_1})^N \Psi_{\boldsymbol{\gamma}+i\omega_1\boldsymbol{e}_j,\varepsilon}\right\} \prod_{s\neq j} \frac{z - e^{\frac{2\pi\gamma_s}{\omega_2}} z^{-1}}{e^{\frac{\pi\gamma_j}{\omega_2}} - e^{\frac{2\pi\gamma_s}{\omega_2}} e^{-\frac{\pi\gamma_j}{\omega_2}}}\,, \end{gathered} \tag{3.60}$$

$$\begin{gathered} \widetilde{t}_N(z)\Psi_{\boldsymbol{\gamma},\varepsilon} = \left(z - z^{-1} e^{\frac{2\pi}{\omega_1}\left(\varepsilon - \sum\limits_{m=1}^{N-1}\gamma_m\right)}\right) \prod_{j=1}^{N-1}\left(z - z^{-1} e^{\frac{2\pi\gamma_j}{\omega_1}}\right)\Psi_{\boldsymbol{\gamma},\varepsilon} + \\ e^{\frac{\pi\varepsilon}{\omega_1}} \sum_{j=1}^{N-1}\left\{(-ig^{\omega_2})^N \Psi_{\boldsymbol{\gamma}-i\omega_2\boldsymbol{e}_j,\varepsilon} + (ig^{\omega_2})^N \Psi_{\boldsymbol{\gamma}+i\omega_2\boldsymbol{e}_j,\varepsilon}\right\} \prod_{s\neq j} \frac{z - e^{\frac{2\pi\gamma_s}{\omega_1}} z^{-1}}{e^{\frac{\pi\gamma_j}{\omega_1}} - e^{\frac{2\pi\gamma_s}{\omega_1}} e^{-\frac{\pi\gamma_j}{\omega_1}}}\,. \end{gathered} \tag{3.61}$$

Using this proposition, one can prove that integral representation (3.47) is still valid for the wave functions of *periodic* q-Toda chain. Namely, the l.h.s. of (3.47) should be considered as the wave function of periodic chain (i.e. as the common eigenfunction for the operators $t_N(z)$ and $\widetilde{t}_N(z)$) with the following changes. The Fourier coefficient $\mathcal{Q}(\boldsymbol{\gamma} \mid \boldsymbol{\lambda})$ factorizes now into the product

$$\mathcal{Q}(\boldsymbol{\gamma} \mid \boldsymbol{\lambda}) = \prod_{j=1}^{N-1} Q(\gamma_j \mid \boldsymbol{\lambda}), \tag{3.62}$$

where $Q(\gamma|\boldsymbol{\lambda})$ is an entire function with an appropriate asymptotic behaviour which satisfies the following system of the mutually dual Baxter equations:

$$\left(\frac{g^{\omega_1}}{2}\right)^{-N}\prod_{k=1}^{N}\sinh\frac{\pi}{\omega_2}(\gamma-\lambda_k)Q(\gamma|\boldsymbol{\lambda}) = i^N Q(\gamma+i\omega_1|\boldsymbol{\lambda}) + i^{-N}Q(\gamma-i\omega_1|\boldsymbol{\lambda}), \tag{3.63a}$$

$$\left(\frac{g^{\omega_2}}{2}\right)^{-N}\prod_{k=1}^{N}\sinh\frac{\pi}{\omega_1}(\gamma-\lambda_k)Q(\gamma|\boldsymbol{\lambda}) = i^N Q(\gamma+i\omega_2|\boldsymbol{\lambda}) + i^{-N}Q(\gamma-i\omega_2|\boldsymbol{\lambda}). \tag{3.63b}$$

(compare with (3.44)).

Remark 3.3 *In the limit $\omega_2 \to \infty$ the equation (3.63a) goes to the standard Baxter equation for the N-particle periodic Toda chain [32] with Planck constant $\hbar = \omega_1$ provided $g^{\omega_1} = \frac{2\pi}{\omega_2}[1 + O(\omega_2^{-1})]$.*

We would like to stress that the proper asymptotic behaviour of the solution of Baxter equations is fixed by the condition that the integral (3.47) converges. Together with analyticity condition for the solution this leads to the quantization conditions of Gutzwiller's type for the eigenvalues $\boldsymbol{\lambda}$ [33], [35], [36]. Note that the Baxter equation (3.63a) has been obtained in [37] in the framework of separation of variables [32]. A similar *system* of Baxter equations (3.63) appeared for the first time in [11], [38] in the models different from ours, but with the same type of duality property.

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A Double sine function

We give here a short summary of the properties of the double sines and related functions. The theory of double sines, double gamma functions, etc. goes back to the papers of Barnes

[12], [13], with more recent additions of Shintani [15] and Kurokawa [16]; the closely related quantum dilogarithms were independently introduced by Faddeev and Kashaev [17] in connection with the Quantum Inverse Scattering Method. See also relevant items on the subject in [39], [40].

A.1 Definition and main properties

The basic properties of double sines listed below are extracted mainly from [15] and [16]; all main ideas are contained already in the papers of Barnes [12]-[14] [9] .

I. Integral formulae. Set $\boldsymbol{\omega} = (\omega_1, \omega_2)$, $\omega_1, \omega_2 > 0$. The double sine $S_2(z|\boldsymbol{\omega})$ is defined by the integral [12], [15]

$$\log S_2(z|\boldsymbol{\omega}) = \int\limits_{\mathcal{C}_H} \frac{\sinh(z - \frac{\omega_1+\omega_2}{2})t}{2\sinh\frac{\omega_1 t}{2}\sinh\frac{\omega_2 t}{2}} \log(-t)\,\frac{dt}{2\pi i t} \tag{A.1}$$

in the region

$$0 < \operatorname{Re} z < \omega_1+\omega_2, \tag{A.2}$$

where the contour $\mathcal{C}_H$ is drawn on Fig. 1:

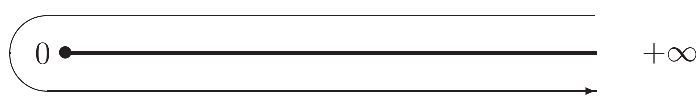

Fig.1 The Hankel contour $\mathcal{C}_H$

An equivalent integral representation is easily derived from (A.1):

$$\log S_2(z|\boldsymbol{\omega}) = \frac{\pi i}{2}\, B_{2,2}(z|\boldsymbol{\omega}) + \int\limits_{\mathbb{R}+i0} \frac{e^{zt}}{(e^{\omega_1 t}-1)(e^{\omega_2 t}-1)}\,\frac{dt}{t}, \tag{A.3}$$

where

$$B_{2,2}(z|\boldsymbol{\omega}) = \frac{z^2}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{\omega_1\omega_2}\,z + \frac{\omega_1^2 + 3\omega_1\omega_2 + \omega_2^2}{6\omega_1\omega_2} \tag{A.4}$$

and the contour is drawn on Fig. 2:

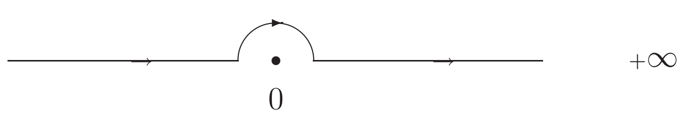

Fig.2 Contour $\mathbb{R} + i0$

[9] In [12, 13] Barnes have developed the complete theory of the so called double gamma functions $\Gamma_2(z|\omega_1, \omega_2)$. The double sine function appeared for the first time in paper by Shintani [15] as a ratio of two appropriate double gamma functions.

Note that

$$B_{2,2}(\omega_1+\omega_2-z|\boldsymbol{\omega}) = B_{2,2}(z|\boldsymbol{\omega})\,. \tag{A.5}$$

From (A.1) one can also derive that

$$\log S_2(z|\boldsymbol{\omega}) = \int\limits_0^\infty \left\{ \frac{\sinh(z-\frac{\omega_1+\omega_2}{2})t}{2\sinh\frac{\omega_1 t}{2}\sinh\frac{\omega_2 t}{2}} - \frac{1}{\omega_1\omega_2 t}\,(2z-\omega_1-\omega_2)\right\}\frac{dt}{t}\,. \tag{A.6}$$

II. Series and product formulae. Evaluating (A.6) by the residue formula, one obtains the series expansions which are valid in the regions $\operatorname{Im} z > 0$, and $\operatorname{Im} z < 0$, respectively:

$$\log S_2(z|\boldsymbol{\omega}) = \frac{\pi i}{2}\,B_{2,2}(z|\boldsymbol{\omega}) + \sum_{n=1}^\infty \frac{1}{n}\left\{ \frac{e^{\frac{2\pi i n z}{\omega_1}}}{e^{\frac{2\pi i n\omega_2}{\omega_1}}-1} + \frac{e^{\frac{2\pi i n z}{\omega_2}}}{e^{\frac{2\pi i n\omega_1}{\omega_2}}-1}\right\}, \tag{A.7}$$

$$\log S_2(z|\boldsymbol{\omega}) = -\frac{\pi i}{2}\,B_{2,2}(z|\boldsymbol{\omega}) + \sum_{n=1}^\infty \frac{1}{n}\left\{ \frac{e^{-\frac{2\pi i n z}{\omega_1}}}{e^{-\frac{2\pi i n\omega_2}{\omega_1}}-1} + \frac{e^{-\frac{2\pi i n z}{\omega_2}}}{e^{-\frac{2\pi i n\omega_1}{\omega_2}}-1}\right\}. \tag{A.8}$$

By an appropriate deformation of the contour $\mathcal{C}_H$ the double sine (A.1) can be extended to all complex values of $\omega_1,\ \omega_2$, provided that $\frac{\omega_1}{\omega_2}\notin\mathbb{R}_-$ [12]. Namely, the contour is in general along the bisector of the smallest angle between $-\arg\omega_1$ and $-\arg\omega_2$ enclosing the origin but no other poles of the integrand in (A.1) (the contour $\mathbb{R}+i0$ should be rotated similarly). As a corollary, we obtain the product expansions which are valid when $\operatorname{Im}\frac{\omega_1}{\omega_2}>0$):

$$\begin{aligned} S_2(z|\boldsymbol{\omega}) \;&=\; e^{\frac{\pi i}{2}B_{2,2}(z|\boldsymbol{\omega})}\,\frac{\prod\limits_{m=0}^\infty\Big(1-q^{2m}e^{\frac{2\pi i z}{\omega_2}}\Big)}{\prod\limits_{m=1}^\infty\Big(1-\widetilde{q}^{\,-2m}e^{\frac{2\pi i z}{\omega_1}}\Big)} \;= \\ &e^{-\frac{\pi i}{2}B_{2,2}(z|\boldsymbol{\omega})}\,\frac{\prod\limits_{m=0}^\infty\Big(1-\widetilde{q}^{\,-2m}e^{-\frac{2\pi i z}{\omega_1}}\Big)}{\prod\limits_{m=1}^\infty\Big(1-q^{2m}e^{-\frac{2\pi i z}{\omega_2}}\Big)}\,, \end{aligned} \tag{A.9}$$

where

$$q = e^{\pi i\frac{\omega_1}{\omega_2}}\,, \qquad \widetilde{q} = e^{\pi i\frac{\omega_2}{\omega_1}}\,.$$

The equality of the two expressions is due to the modular transformation law for the theta function $\theta_1(z|\frac{\omega_1}{\omega_2})$.

III. Functional relations. The function $S_2(z|\boldsymbol{\omega})$ satisfies the difference equations

$$\frac{S_2(z+\omega_1|\boldsymbol{\omega})}{S_2(z|\boldsymbol{\omega})} = \frac{1}{2\sin\frac{\pi z}{\omega_2}}\,, \tag{A.10a}$$

$$\frac{S_2(z+\omega_2|\boldsymbol{\omega})}{S_2(z|\boldsymbol{\omega})} = \frac{1}{2\sin\frac{\pi z}{\omega_1}}\,. \tag{A.10b}$$

Moreover,

$$\begin{aligned} S_2(z|\boldsymbol{\omega})S_2(-z|\boldsymbol{\omega}) &= -4\sin\frac{\pi z}{\omega_1}\sin\frac{\pi z}{\omega_2}\,, \\ S_2(z|\boldsymbol{\omega})S_2(\omega_1+\omega_2-z|\boldsymbol{\omega}) &= 1\,. \end{aligned} \tag{A.11}$$

IV. Poles and zeroes. The zeroes and poles of $S_2(z|\boldsymbol{\omega})$ are as follows:

$$\begin{aligned} &\text{poles at} \quad z = n_1\omega_1 + n_2\omega_2, \quad n_1,\ n_2 \geq 1, \\ &\text{zeros at} \quad z = n_1\omega_1 + n_2\omega_2, \quad n_1, n_2 \leq 0. \end{aligned} \tag{A.12}$$

Moreover,

$$\lim_{z\to 0} z^{-1} S_2(z|\boldsymbol{\omega}) = \frac{2\pi}{\sqrt{\omega_1\omega_2}}\,. \tag{A.13}$$

Hence, from (A.10) and (A.13),

$$S_2(\omega_1|\boldsymbol{\omega}) \ = \ \sqrt{\frac{\omega_2}{\omega_1}}\,, \quad S_2(\omega_2|\boldsymbol{\omega}) \ = \ \sqrt{\frac{\omega_1}{\omega_2}}\,. \tag{A.14}$$

Using (A.10), (A.13), one can calculate the residues of $S_2(z|\boldsymbol{\omega})$ and $S_2^{-1}(z|\boldsymbol{\omega})$ at the corresponding points (A.12):

$$\lim_{z\to 0} zS_2(z+n_1\omega_1+n_2\omega_2|\boldsymbol{\omega}) = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\, \frac{(-1)^{n_1 n_2}}{\prod\limits_{k=1}^{n_1-1} 2\sin\frac{\pi k\omega_1}{\omega_2} \prod\limits_{m=1}^{n_2-1} 2\sin\frac{\pi m\omega_2}{\omega_1}}, \tag{A.15a}$$

$$\lim_{z\to 0} zS_2^{-1}(z-n_1\omega_1-n_2\omega_2|\boldsymbol{\omega}) = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\, \frac{(-1)^{n_1 n_2+n_1+n_2}}{\prod\limits_{k=1}^{n_1} 2\sin\frac{\pi k\omega_1}{\omega_2} \prod\limits_{m=1}^{n_2} 2\sin\frac{\pi m\omega_2}{\omega_1}}. \tag{A.15b}$$

V. Asymptotics. Assuming that $\operatorname{Im}\frac{\omega_1}{\omega_2} > 0$, we have

$$S_2(z|\boldsymbol{\omega}) \underset{z\to\infty}{\longrightarrow} \begin{cases} e^{\frac{\pi i}{2}B_{2,2}(z|\boldsymbol{\omega})}, & \arg\omega_1 < \arg z < \arg\omega_2 + \pi, \\ e^{-\frac{\pi i}{2}B_{2,2}(z|\boldsymbol{\omega})}, & \arg\omega_1 - \pi < \arg z < \arg\omega_2, \\ \dfrac{e^{-\frac{\pi i}{2}B_{2,2}(z|\boldsymbol{\omega})}}{\prod\limits_{m=1}^{\infty}\left(1 - q^{2m}e^{-\frac{2\pi i z}{\omega_2}}\right)}, & \arg\omega_2 < \arg z < \arg\omega_1, \\ e^{\frac{\pi i}{2}B_{2,2}(z|\boldsymbol{\omega})} \prod\limits_{m=0}^{\infty}\left(1 - q^{2m}e^{\frac{2\pi i z}{\omega_2}}\right), & \arg\omega_2+\pi < \arg z < \arg\omega_1+\pi. \end{cases}$$

VI. Complex conjugation.

$$\overline{S_2(z|\boldsymbol{\omega})} = S_2(\overline{z}\,|\overline{\boldsymbol{\omega}})\,. \tag{A.16}$$

A.2 Related functions

It is convenient to introduce the function $\mathcal{S}(z|\boldsymbol{\omega})$ as follows:

$$S_2(z|\boldsymbol{\omega}) = e^{\frac{\pi i}{2} B_{2,2}(z|\boldsymbol{\omega})} \mathcal{S}(z|\boldsymbol{\omega}), \tag{A.17}$$

where according to (A.3)

$$\log \mathcal{S}(z|\boldsymbol{\omega}) \ = \int\limits_{\mathbb{R}+i0} \frac{e^{zt}}{(e^{\omega_1 t}-1)(e^{\omega_2 t}-1)} \frac{dt}{t}. \tag{A.18}$$

For complex periods with $\operatorname{Im} \frac{\omega_1}{\omega_2} > 0$ we get

$$\mathcal{S}(z|\boldsymbol{\omega}) \ = \ \frac{\prod\limits_{m=0}^{\infty} \left(1 - q^{2m} e^{\frac{2\pi i z}{\omega_2}}\right)}{\prod\limits_{m=1}^{\infty} \left(1 - \widetilde{q}^{\,-2m} e^{\frac{2\pi i z}{\omega_1}}\right)} \ = \ e^{-\pi i B_{2,2}(z|\boldsymbol{\omega})} \ \frac{\prod\limits_{m=0}^{\infty} \left(1 - \widetilde{q}^{\,-2m} e^{-\frac{2\pi i z}{\omega_1}}\right)}{\prod\limits_{m=1}^{\infty} \left(1 - q^{2m} e^{-\frac{2\pi i z}{\omega_2}}\right)}. \tag{A.19}$$

Evidently, $\mathcal{S}(z|\boldsymbol{\omega})$ satisfies the difference equations

$$\frac{\mathcal{S}(z+\omega_1|\boldsymbol{\omega})}{\mathcal{S}(z|\boldsymbol{\omega})} = \frac{1}{1 - e^{\frac{2\pi i z}{\omega_2}}}, \tag{A.20a}$$

$$\frac{\mathcal{S}(z+\omega_2|\boldsymbol{\omega})}{\mathcal{S}(z|\boldsymbol{\omega})} = \frac{1}{1 - e^{\frac{2\pi i z}{\omega_1}}} \tag{A.20b}$$

and obeys the condition

$$\overline{S(z|\boldsymbol{\omega})} = S^{-1}(\omega_1 + \omega_2 - \overline{z}\,|\boldsymbol{\omega}) \tag{A.21}$$

when ω_1, ω_2 are real or $\overline{\omega}_1 = \omega_2$.

Let us finally mention that the *quantum dilogarithm* $e_{\boldsymbol{\omega}}(z)$ defined in [10], [17] is related to $\mathcal{S}(z|\boldsymbol{\omega})$ via

$$e_{\boldsymbol{\omega}}(z) = \mathcal{S}(\frac{\omega_1+\omega_2}{2} - iz|\boldsymbol{\omega})\,, \tag{A.22}$$

while the *hyperbolic gamma function* $G(z|\boldsymbol{\omega})$ introduced in [28] is

$$G(z|\boldsymbol{\omega}) = S_2(\frac{\omega_1+\omega_2}{2} + iz|\boldsymbol{\omega})\,. \tag{A.23}$$

A.3 Fourier transform

The function $\mathcal{S}(it|\boldsymbol{\omega})$ has the following asymptotics:

$$\mathcal{S}(it|\boldsymbol{\omega}) \underset{t\to\infty}{\longrightarrow} \begin{cases} 1, & \arg\omega_1 - \dfrac{\pi}{2} < \arg t < \arg\omega_2 + \dfrac{\pi}{2} \\ e^{-\pi i B_{2,2}(it|\boldsymbol{\omega})}, & \arg\omega_1 + \dfrac{\pi}{2} < \arg t < \arg\omega_2 + \dfrac{3\pi}{2} \end{cases} \tag{A.24}$$

(we are not interested in behavior in two remaining sectors). Each sector is divided into two subsectors (+) (resp. (−)) where the exponential $e^{\frac{\pi i t^2}{\omega_1\omega_2}}$ (resp. $e^{-\frac{\pi i t^2}{\omega_1\omega_2}}$) is rapidly decreasing. To be more precise, the sectors (+) are determined by inequalities

$$\frac{1}{2}(\arg\omega_1+\arg\omega_2)+\pi<\arg t<\arg\omega_2+\frac{3\pi}{2}\,, \tag{A.25a}$$

$$\frac{1}{2}(\arg\omega_1+\arg\omega_2)<\arg t<\arg\omega_2+\frac{\pi}{2}\,, \tag{A.25b}$$

while the sectors (−) are determined by inequalities

$$\arg\omega_1+\frac{\pi}{2}<\arg t<\frac{1}{2}(\arg\omega_1+\arg\omega_2)+\pi\,, \tag{A.26a}$$

$$\arg\omega_1-\frac{\pi}{2}<\arg t<\frac{1}{2}(\arg\omega_1+\arg\omega_2)\,. \tag{A.26b}$$

(see Fig.3 for the typical complex periods ω_1 and ω_2).

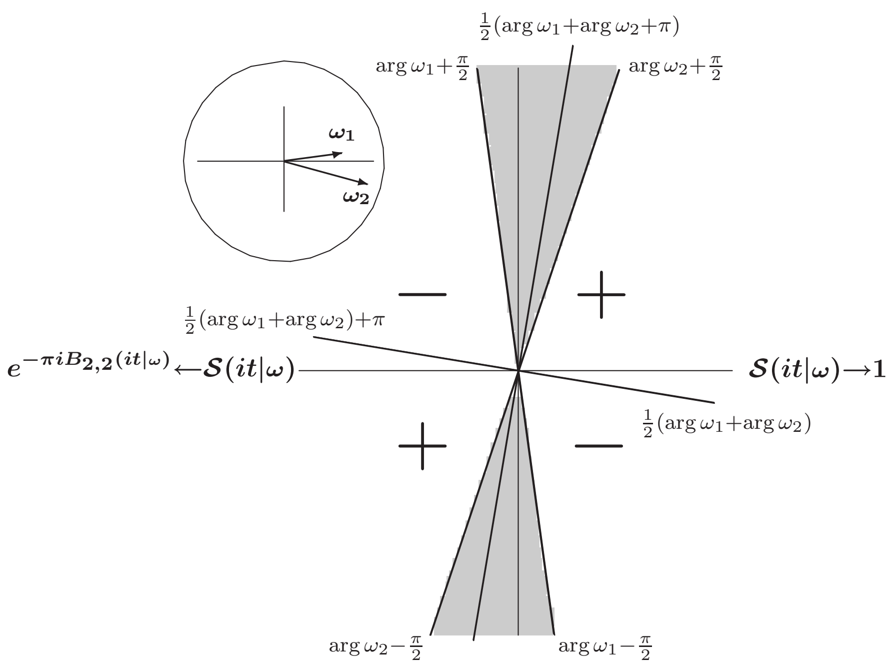

Fig. 3.

There exist remarkable integral formulae with the double sine type functions [11, 21]. In particular, there are the following Fourier transformation formulae [11] [10] :

$$\int_\Gamma \mathcal{S}(it+\omega_1+\omega_2-a)\,e^{\frac{2\pi i z t}{\omega_1\omega_2}}dt=\sqrt{\omega_1\omega_2}\;e^{-\frac{\pi i}{2}B_{2,2}(0)}\;\mathcal{S}^{-1}(-iz)\;e^{\frac{2\pi z a}{\omega_1\omega_2}} \tag{A.27a}$$

[10] Actually, any three formulae in (A.27) are consequences of the fourth one. Nevertheless, we list all of them for convenience.

$$\int\limits_{\mathrm{L}} \mathcal{S}^{-1}(it+a)\, e^{\frac{2\pi izt}{\omega_1\omega_2}} dt = \sqrt{\omega_1\omega_2}\; e^{\frac{\pi i}{2}B_{2,2}(0)}\; \mathcal{S}(\omega_1+\omega_2+iz)\; e^{-\frac{2\pi za}{\omega_1\omega_2}} \tag{A.27b}$$

$$\int\limits_{\Gamma'} \mathcal{S}(-it+\omega_1+\omega_2-a)\, e^{\frac{2\pi izt}{\omega_1\omega_2}} dt = \sqrt{\omega_1\omega_2}\; e^{-\frac{\pi i}{2}B_{2,2}(0)}\; \mathcal{S}^{-1}(iz)\; e^{-\frac{2\pi za}{\omega_1\omega_2}} \tag{A.27c}$$

$$\int\limits_{\mathrm{L}'} \mathcal{S}^{-1}(-it+a)\, e^{\frac{2\pi izt}{\omega_1\omega_2}} dt = \sqrt{\omega_1\omega_2}\; e^{\frac{\pi i}{2}B_{2,2}(0)}\; \mathcal{S}(\omega_1+\omega_2-iz)\; e^{\frac{2\pi za}{\omega_1\omega_2}} \tag{A.27d}$$

The notations here are as follows. The contours Γ and L' are above the poles

$$t_{n_1,n_2} = -i(a+n_1\omega_1+n_2\omega_2) \qquad\qquad (n_1,n_2\geq 0) \tag{A.28}$$

of the integrands in (A.27a) and (A.27d) while the contours L and Γ' are below the poles

$$t'_{n_1,n_2} = i(a+n_1\omega_1+n_2\omega_2) \qquad\qquad (n_1,n_2\geq 0) \tag{A.29}$$

of the integrands in (A.27b) and (A.27c).
Further, the contours Γ and L are beginning in subsectors (A.25a) and (A.26a) respectively, but may lie in the whole sector $[\arg\omega_1-\frac{\pi}{2}\,,\arg\omega_2+\frac{\pi}{2}]$, while the contours Γ' and L' may lie in the whole sector $[\arg\omega_1+\frac{\pi}{2}\,,\arg\omega_2+\frac{3\pi}{2}]$, but are ending in subsectors (A.25b) and (A.26b), respectively.

Provided such a description, the formulae (A.27a) and (A.27b) hold in the region

$$\arg z \notin \left[\arg\omega_2-\frac{\pi}{2}, \arg\omega_1-\frac{\pi}{2}\right], \tag{A.30}$$

while in (A.27c) and (A.27d)

$$\arg z \notin \left[\arg\omega_2+\frac{\pi}{2}, \arg\omega_1+\frac{\pi}{2}\right]. \tag{A.31}$$

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